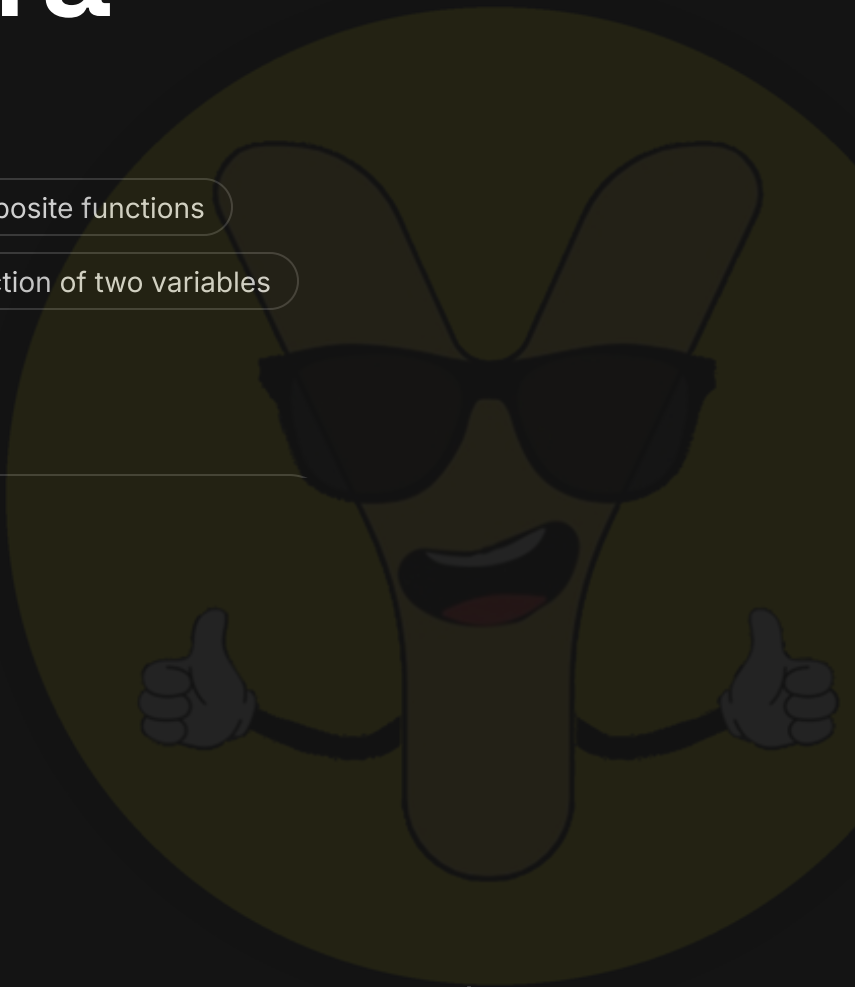
AKTU · 1ST YEAR · SEMESTER 1 · AAS103D

Calculus and Linear Algebra

Unit 1 — Calculus

- Introduction of Limits, continuity and differentiability for function of two variables
- Partial derivatives
- Composite functions
- Total derivative
- Euler's Theorem for homogeneous functions
- Change of variables
- Expansion for function of two variables
- Jacobians
- Properties of Jacobian (without proof)
- Maxima & Minima for a function of several variables
- Lagrange's method of multipliers
- Approximation of Errors



DECODED BY

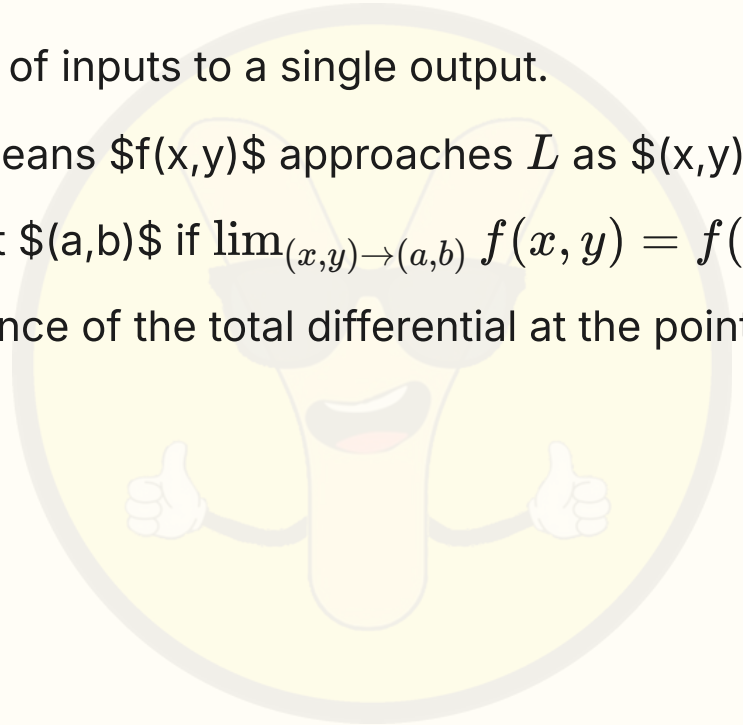
Chirag Jain

Founder, Ycotes · Gold Medalist in Artificial Intelligence, IIT Madras · taught 10,000+ students

Introduction to Multivariable Calculus

Understanding functions of two or more variables is essential for modeling complex systems.

- A function $z = f(x, y)$ maps a pair of inputs to a single output.
- Limits: $\lim_{(x,y)\to(a,b)} f(x, y) = L$ means $f(x,y)$ approaches L as (x,y) nears (a,b) along any path.
- Continuity: $f(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.
- Differentiability: Requires the existence of the total differential at the point.

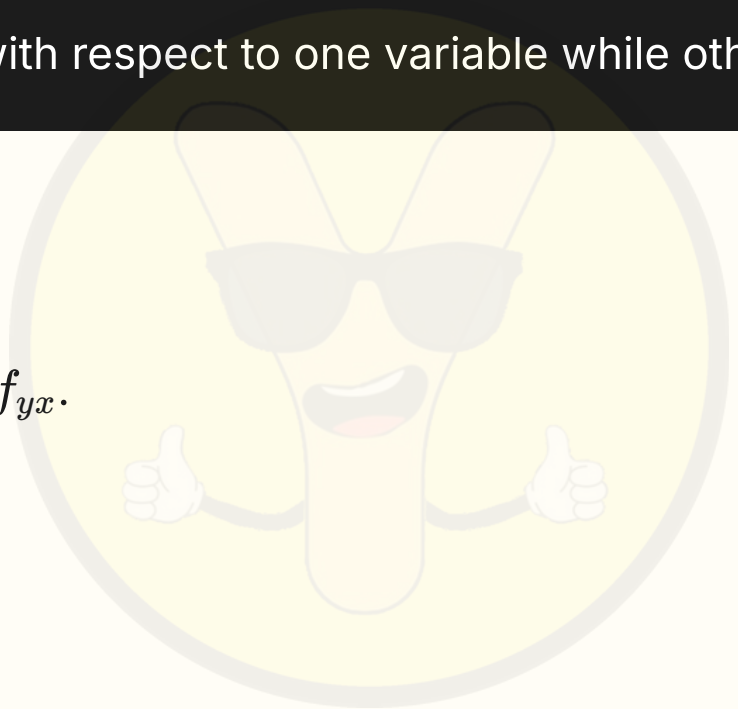


Partial Derivatives

PARTIAL DERIVATIVE

The rate of change of a function with respect to one variable while others are held constant.

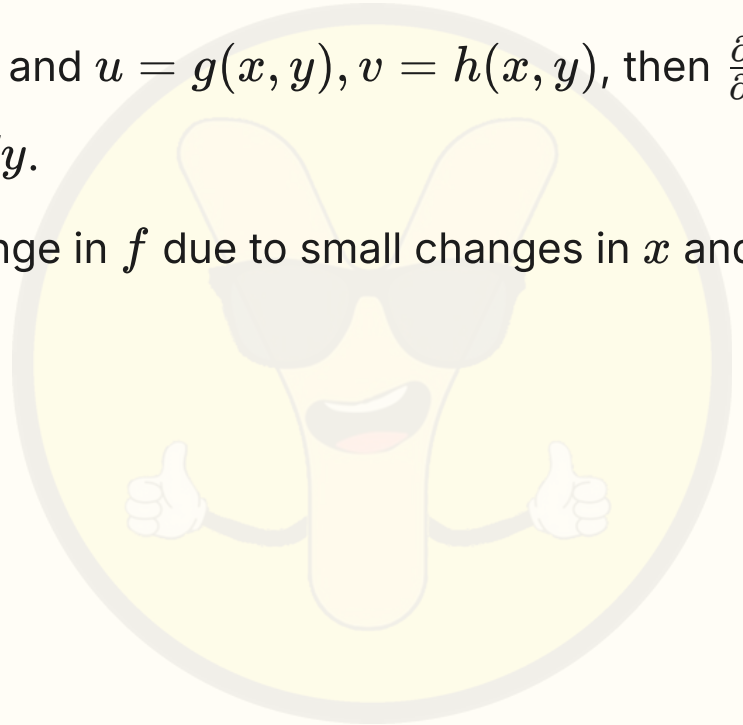
- $\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$
- $\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y}$
- Higher order partials: $f_{xx}, f_{yy}, f_{xy}, f_{yx}$.



Composite and Total Derivatives

Handling functions within functions and total change.

- Composite Function: If $z = f(u, v)$ and $u = g(x, y), v = h(x, y)$, then $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$.
- Total Derivative: $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$.
- This formula captures the total change in f due to small changes in x and y simultaneously.



Euler’s Theorem

HOMOGENEOUS FUNCTION

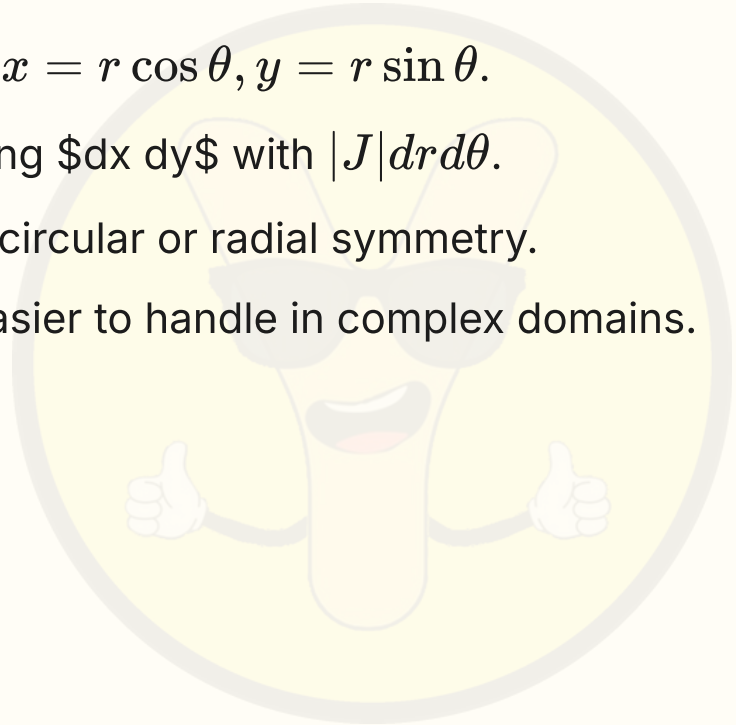
A function $f(x,y)$ is homogeneous of degree n if $f(tx, ty) = t^n f(x, y)$.

- Theorem Statement: If f is homogeneous of degree n, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$.
- Useful for simplifying complex partial differential equations.
- Extension: $x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy} = n(n-1)f$.

Change of Variables

Transforming coordinate systems to simplify integration or differentiation.

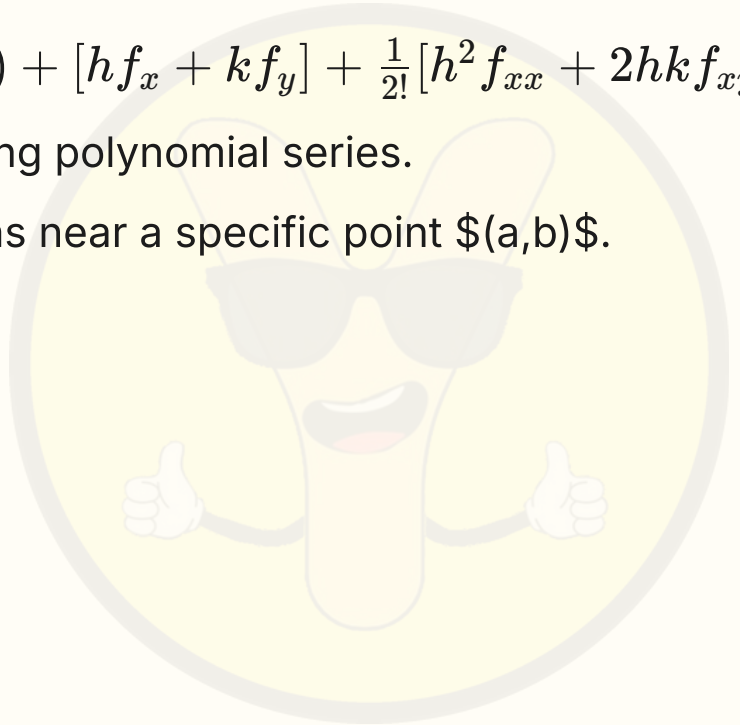
- Commonly used: Polar coordinates $x = r\cos\theta, y = r\sin\theta$.
- The transformation involves replacing $dx dy$ with $|J|drd\theta$.
- Essential for solving problems with circular or radial symmetry.
- Ensures boundary conditions are easier to handle in complex domains.



Expansion for Two Variables

Taylor series expansion for functions of two variables.

- Formula: $f(a+h, b+k) = f(a,b) + [hf_x + kf_y] + \frac{1}{2!}[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}] + \dots$
- Approximates smooth functions using polynomial series.
- Crucial for local analysis of functions near a specific point (a,b).



Jacobians

JACOBIAN DETERMINANT

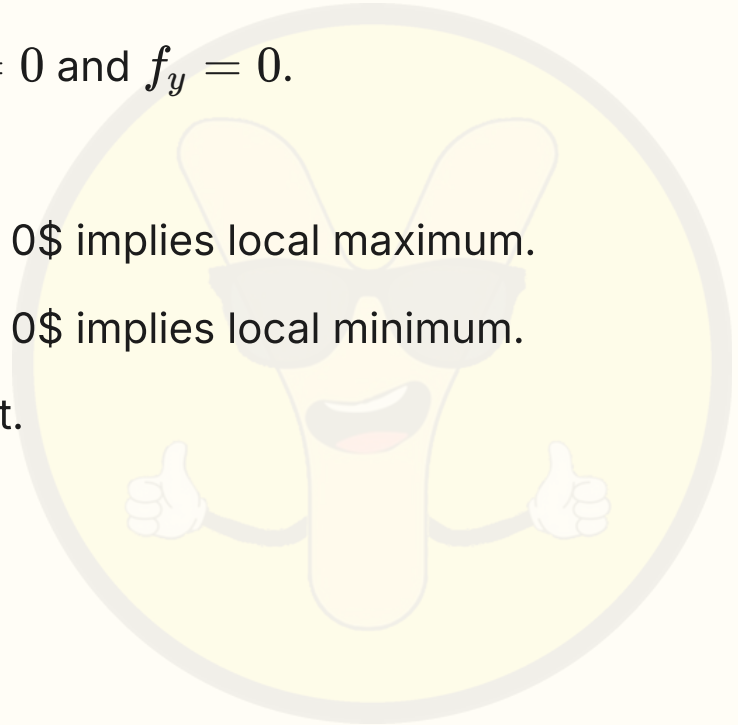
A determinant of partial derivatives representing the local scaling factor of a transformation.

- $J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$
- Properties: If u, v are functionally dependent, $J = 0$.
- Chain rule for Jacobians: $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$.

Maxima and Minima

Finding optimal points for functions of several variables.

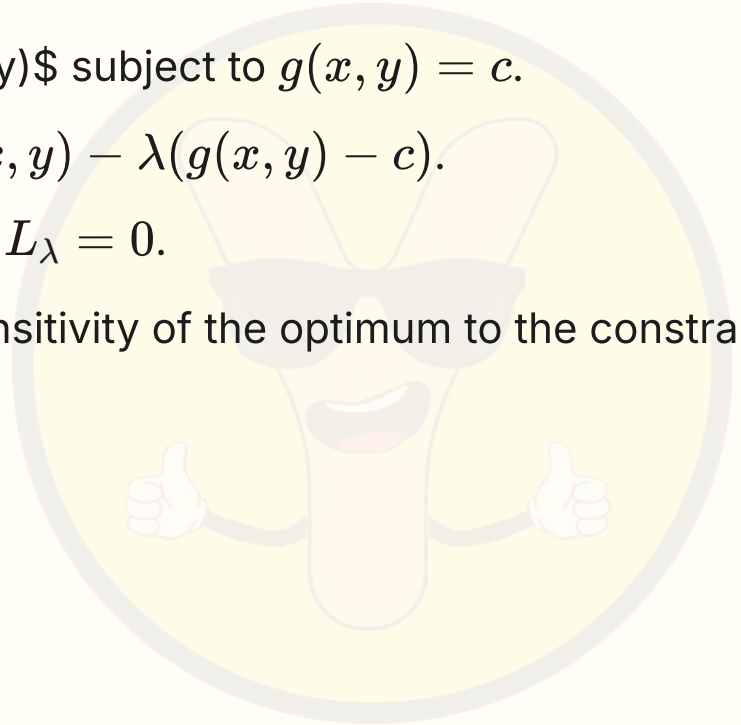
- Stationary points occur where $f_x = 0$ and $f_y = 0$.
- Let $A = f_{xx}, B = f_{xy}, C = f_{yy}$.
- Condition: $AC - B^2 > 0$ and $A < 0$ implies local maximum.
- Condition: $AC - B^2 > 0$ and $A > 0$ implies local minimum.
- If $AC - B^2 < 0$, it is a saddle point.



Lagrange's Method of Multipliers

Optimizing a function subject to equality constraints.

- Objective: Maximize/Minimize $f(x,y)$ subject to $g(x, y) = c$.
- Algorithm: Define $L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$.
- Solve the system: $L_x = 0, L_y = 0, L_\lambda = 0$.
- The parameter λ represents the sensitivity of the optimum to the constraint.

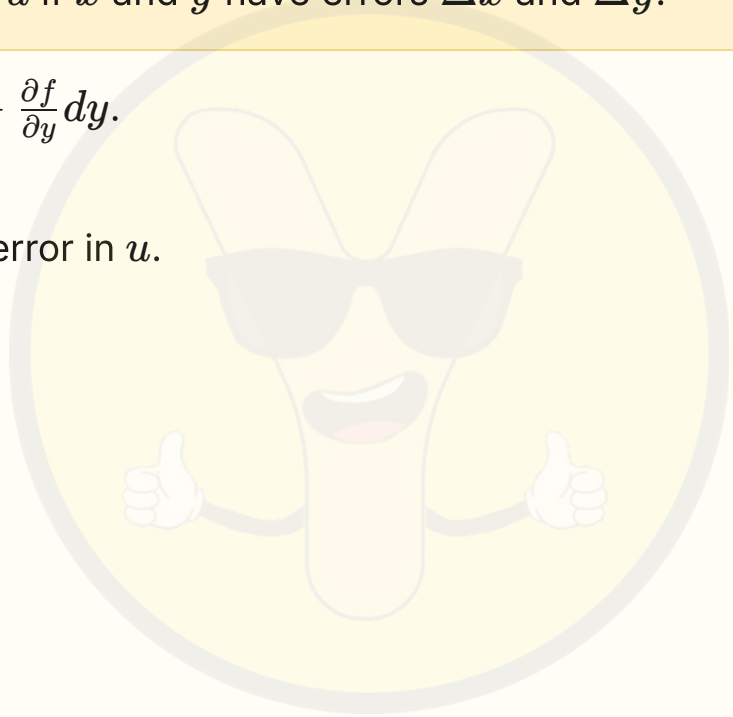


Approximation of Errors

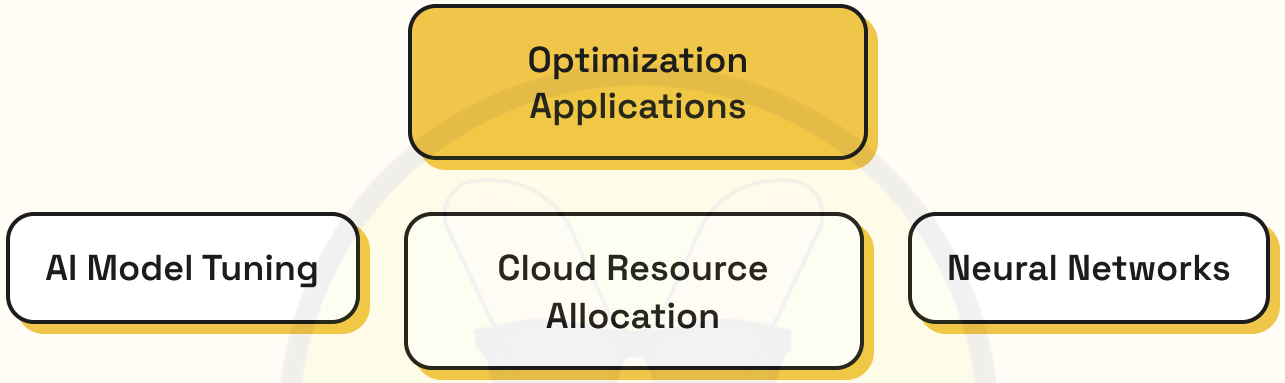
Given $u = f(x, y)$, estimate the error in u if x and y have errors Δx and Δy.

1. Calculate total differential: $du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$.
2. Substitute $dx = \Delta x$ and $dy = \Delta y$.
3. The result du provides the approximate error in u.

$$\Delta u \approx |f_x| \Delta x + |f_y| \Delta y$$



Applications in Modern Systems

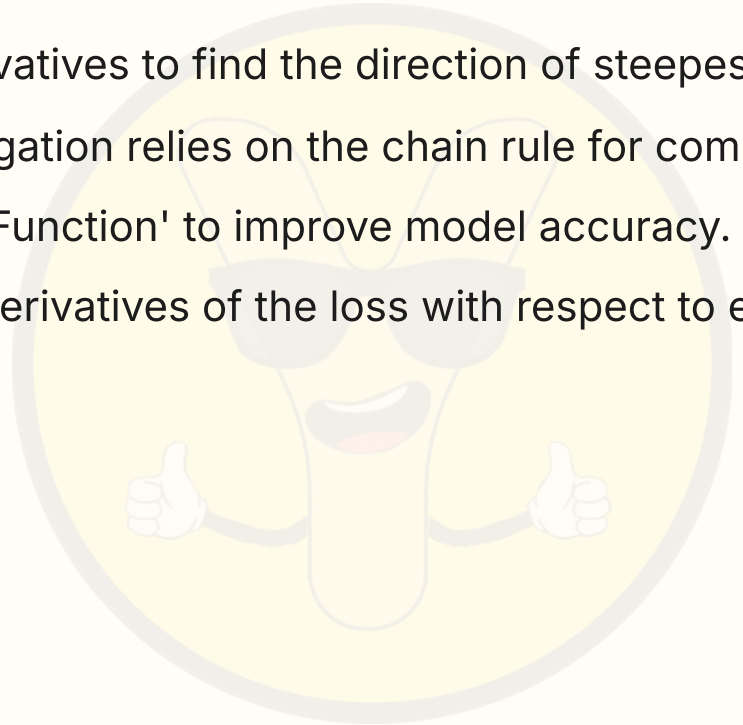


Calculus forms the mathematical backbone for these computational fields.

AI and Neural Network Optimization

How calculus drives learning.

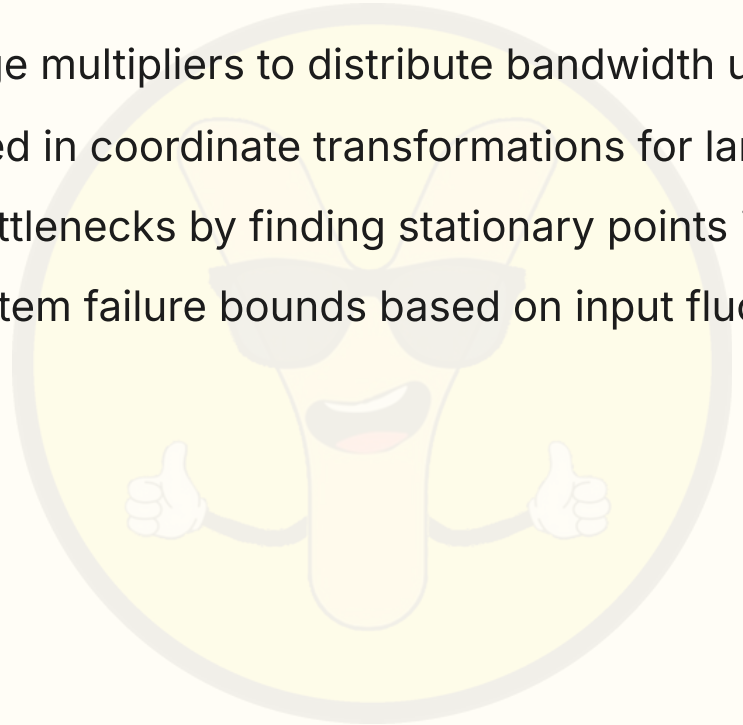
- Gradient Descent: Uses partial derivatives to find the direction of steepest descent.
- Neural Network Tuning: Backpropagation relies on the chain rule for composite functions.
- Optimization: Minimizing the 'Loss Function' to improve model accuracy.
- Weights are updated using partial derivatives of the loss with respect to each weight.



Cloud Computing and Data Systems

Performance and resource management.

- Resource Allocation: Using Lagrange multipliers to distribute bandwidth under constraints.
- Data Processing: Jacobians are used in coordinate transformations for large-scale data mapping.
- Performance Tuning: Identifying bottlenecks by finding stationary points in latency functions.
- Error Approximation: Predicting system failure bounds based on input fluctuations.



Exam corner

Q1 State and prove Euler's theorem for a homogeneous function of two variables.

Q2 Find the local maxima and minima of $f(x, y) = x^3 + y^3 - 3axy$.

Q3 Use Lagrange's method to find the dimensions of a rectangular box of maximum volume with a fixed surface area.

Q4 Define Jacobian and show that if u, v are functions of x, y, then $J(u, v) \cdot J(x, y) = 1$.

Q5 Expand $e^x \sin y$ in powers of x and y up to third degree terms.

Q6 Explain how partial derivatives are utilized in gradient-based machine learning optimization.

Unit 1 done. You are exam-ready.

Revise the Exam corner once more, then open Unit 2 on theycotes.com. Every AKTU subject, unit by unit, free. Notes by Chirag Jain, Ycotes.

