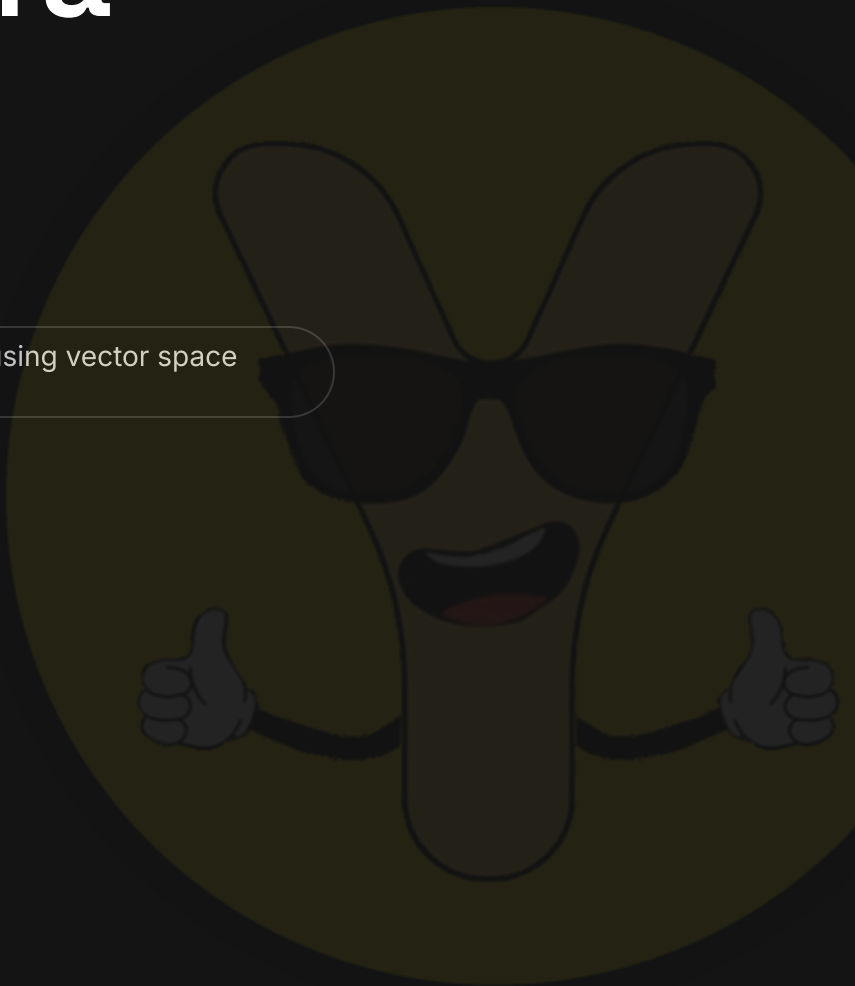
AKTU · 1ST YEAR · SEMESTER 1 · AAS103D

Calculus and Linear Algebra

Unit 5 — Introduction to Vector Space

- Definition of Field
- Introduction to Vector Spaces
- Basic Properties of Vector Spaces
- Sub spaces
- Linear dependence and independence of vectors
- Basis and Dimension
- Application: Engineering Data Modeling and Compression, Signal representation and processing, Noise reduction using vector space concepts



DECODED BY

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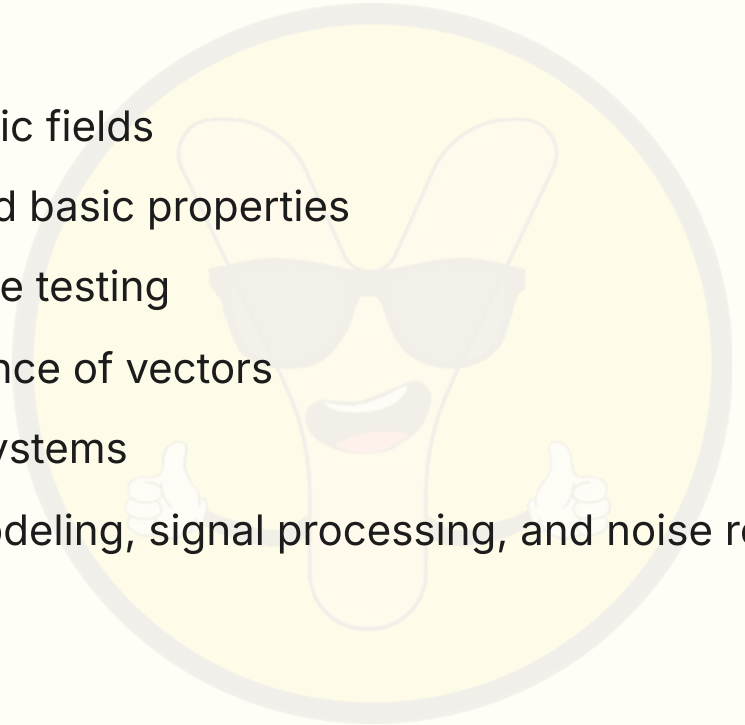
Unit 5: Introduction to Vector Space

Comprehensive coverage of algebraic structures, vector spaces, subspaces, and engineering applications.

- Definition and properties of algebraic fields
- Vector space definition, axioms, and basic properties
- Subspaces and criteria for subspace testing
- Linear dependence and independence of vectors
- Basis, dimension, and coordinate systems
- Engineering applications in data modeling, signal processing, and noise reduction



Definition of a Field

FIELD $(F, +, \cdot)$

A set F equipped with two binary operations, addition $(+)$ and multiplication $(\cdot)$, satisfying axioms of associativity, commutativity, distributivity, identities, and inverses for non-zero elements.

- Must contain at least two distinct elements: additive identity 0 and multiplicative identity 1.
- Common examples used in linear algebra: Field of Real numbers $\mathbb{R}$ and Complex numbers $\mathbb{C}$.
- Rational numbers $\mathbb{Q}$ form a field, but integers $\mathbb{Z}$ do not form a field because multiplicative inverses fail.

Field Axioms Summary

Axiom Property	Addition $(+)$	Multiplication $(\cdot)$
Closure	$a + b \in F$	$a \cdot b \in F$
Associativity	$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Commutativity	$a + b = b + a$	$a \cdot b = b \cdot a$
Identity	$a + 0 = a$	$a \cdot 1 = a$
Inverse	$a + (-a) = 0$	$a \cdot a^{-1} = 1$ (for $a \neq 0$)
Distributivity	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	Applies across both operations

Introduction to Vector Spaces

VECTOR SPACE

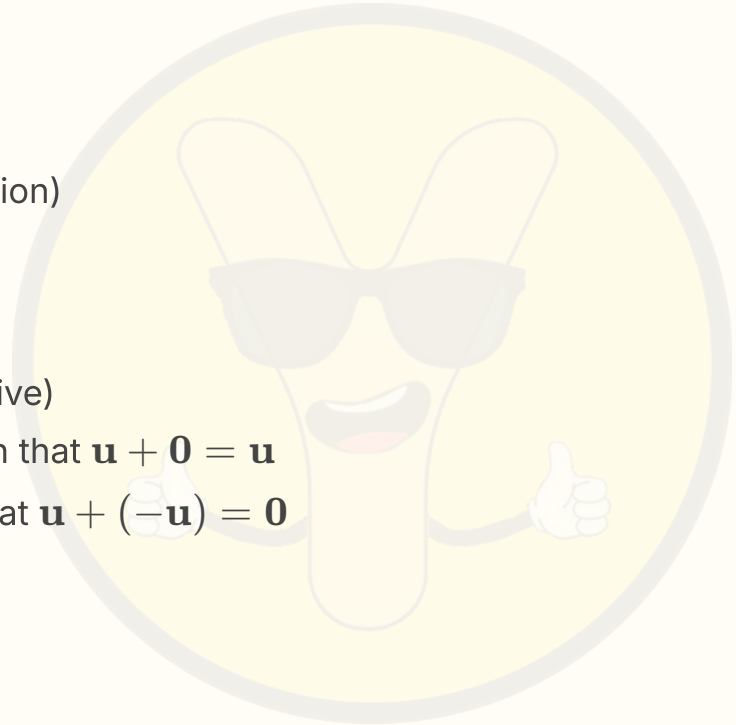
A non-empty set V of objects called vectors, along with a field F of scalars, satisfying two operations: vector addition and scalar multiplication, subject to ten specific axioms.

- Elements of V are vectors denoted as $\mathbf{u}, \mathbf{v}, \mathbf{w}$.
- Elements of F are scalars denoted as $c, d \in \mathbb{R}$ or $\mathbb{C}$.
- Vector addition: $+ : V \times V \to V$.
- Scalar multiplication: $\cdot : F \times V \to V$.

Vector Space Axioms

A set V is a vector space over field F if the following hold for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalars $c, d \in F$:

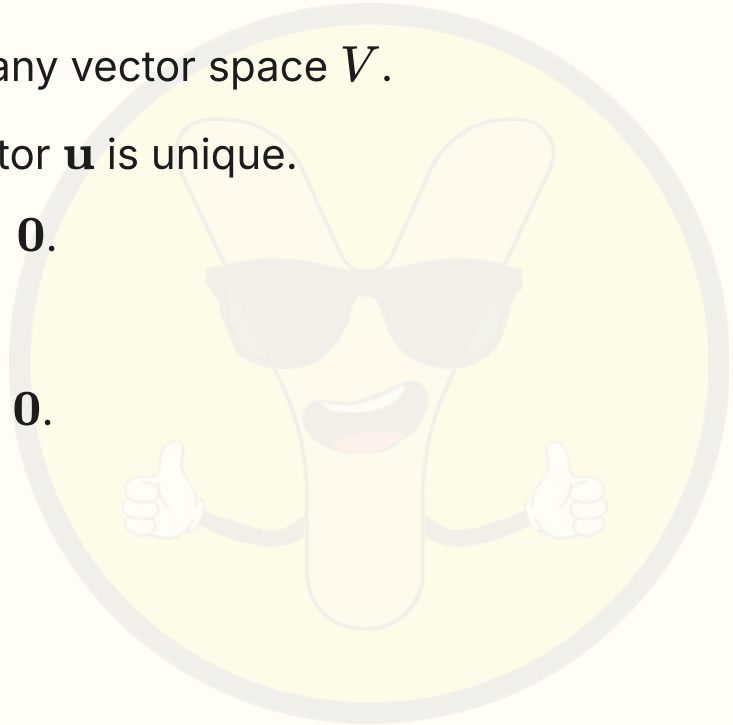
- Closure Axioms
  - $\mathbf{u} + \mathbf{v} \in V$ (Closure under addition)
  - $c\mathbf{u} \in V$ (Closure under scalar multiplication)
- Addition Axioms
  - $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutative)
  - $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (Associative)
  - Existence of additive identity $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
  - Existence of additive inverse $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- Scalar Multiplication Axioms
  - $c(d\mathbf{u}) = (cd)\mathbf{u}$
  - $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
  - $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
  - $1\mathbf{u} = \mathbf{u}$



Basic Properties of Vector Spaces

Several fundamental algebraic properties can be derived directly from the ten vector space axioms:

- The additive identity $\mathbf{0}$ is unique in any vector space V.
- The additive inverse $-\mathbf{u}$ of any vector $\mathbf{u}$ is unique.
- For any scalar c and vector $\mathbf{u}$, $c\mathbf{0} = \mathbf{0}$.
- For any scalar c, $0\mathbf{u} = \mathbf{0}$.
- If $c\mathbf{u} = \mathbf{0}$, then either $c = 0$ or $\mathbf{u} = \mathbf{0}$.
- For any vector $\mathbf{u}$, $(-1)\mathbf{u} = -\mathbf{u}$.



Subspaces

VECTOR SUBSPACE

A subset W of a vector space V is called a subspace of V if W is itself a vector space under the same addition and scalar multiplication operations defined on V.

- Every subspace must contain the zero vector $\mathbf{0}$ of V.
- Subspace test reduces ten axioms to three checks.
- Trivial subspaces of any vector space V are $\{\mathbf{0}\}$ and V itself.

Subspace Testing Criteria

To prove a non-empty subset $W \subseteq V$ is a subspace, verify the following three conditions:

- Step 1: Check if the zero vector of V is in W. (i.e., $\mathbf{0} \in W$).
- Step 2: Closure under addition: For every $\mathbf{u}, \mathbf{v} \in W$, the sum $\mathbf{u} + \mathbf{v} \in W$.
- Step 3: Closure under scalar multiplication: For every $\mathbf{u} \in W$ and scalar c, $c\mathbf{u} \in W$.
- Alternatively, use the single-condition test: $c\mathbf{u} + d\mathbf{v} \in W$ for all $\mathbf{u}, \mathbf{v} \in W$ and scalars c, d.

Subspace Verification Example

Determine if the subset $W = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

1. Step 1: Zero vector check. Setting $x = 0, y = 0$ gives $(0, 0, 0) \in W$. Zero vector exists.
2. Step 2: Addition closure. Let $\mathbf{u} = (x_1, y_1, 0)$ and $\mathbf{v} = (x_2, y_2, 0)$. Sum is $\mathbf{u} + \mathbf{v} = (x_1 + x_2, y_1 + y_2, 0)$, which has a third coordinate 0, so $\mathbf{u} + \mathbf{v} \in W$.
3. Step 3: Scalar multiplication closure. Let $c \in \mathbb{R}$. Then $c(x_1, y_1, 0) = (cx_1, cy_1, 0)$, which has a third coordinate 0, so $c\mathbf{u} \in W$.

Since all three conditions are satisfied, W is a valid subspace of $\mathbb{R}^3$.

Linear Dependence and Independence

LINEAR INDEPENDENCE

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is linearly independent if the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ has only the trivial solution $c_1 = c_2 = \cdots = c_n = 0$.

- If there exist scalars, not all zero, satisfying the equation, the vectors are linearly dependent.
- Linearly dependent sets contain redundant vectors that can be expressed as linear combinations of others.
- Any set containing the zero vector is automatically linearly dependent.

Testing Linear Independence

Algorithm to determine if a set of vectors in $\mathbb{R}^n$ is linearly independent:

- Step 1: Form a matrix A whose columns are the given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
- Step 2: Reduce matrix A to row echelon form using elementary row operations.
- Step 3: Count the number of pivot columns. If every column is a pivot column, the vectors are linearly independent.
- Step 4: If there is at least one free variable (non-pivot column), non-trivial solutions exist, and vectors are linearly dependent.

Linear Independence Problem

Determine if vectors $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (0, 1, 4)$, and $\mathbf{v}_3 = (2, 5, 10)$ are linearly independent.

1. Step 1: Set up the linear combination equation $c_1(1, 2, 3) + c_2(0, 1, 4) + c_3(2, 5, 10) = (0, 0, 0)$.
2. Step 2: Form matrix A with vectors as columns and compute determinant: $\det(A) = 1(10 - 20) - 0 + 2(8 - 3) = -10 + 10 = 0$.
3. Step 3: Since the determinant is zero, the columns are linearly dependent.

The vectors are linearly dependent. Specifically, $2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_3$.

Basis and Dimension

BASIS OF A VECTOR SPACE

A subset $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of a vector space V is called a basis for V if: (1) B is linearly independent, and (2) B spans V.

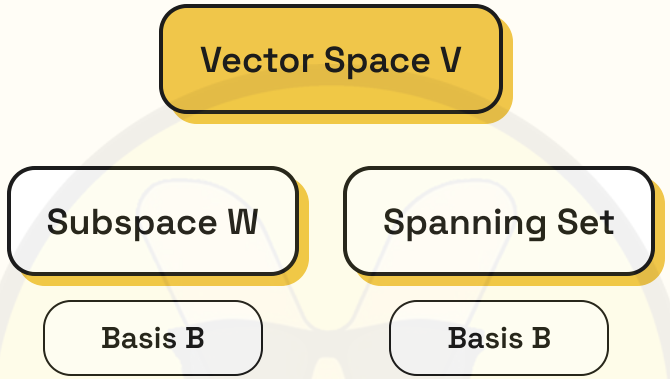
- Spanning means every vector in V can be written as a linear combination of vectors in B.
- Dimension of V is defined as the number of vectors in any basis of V, denoted as $\dim(V) = n$.
- If a vector space requires infinitely many basis vectors, it is infinite-dimensional.

Properties of Basis and Dimension

Fundamental theorems governing bases and dimensions in linear algebra:

- Every vector space (except the zero space) possesses a basis.
- Any two bases of a vector space V contain the exact same number of vectors.
- If $\dim(V) = n$, then any set of more than n vectors in V is linearly dependent.
- If $\dim(V) = n$, then any set of fewer than n vectors cannot span V.
- Standard basis for $\mathbb{R}^n$ consists of unit vectors $\mathbf{e}_1 = (1, 0, \ldots, 0), \ldots, \mathbf{e}_n = (0, \ldots, 0, 1)$.

Vector Space Hierarchy



Hierarchical relationship from a general vector space down to its fundamental dimension.

Engineering Application: Data Modeling

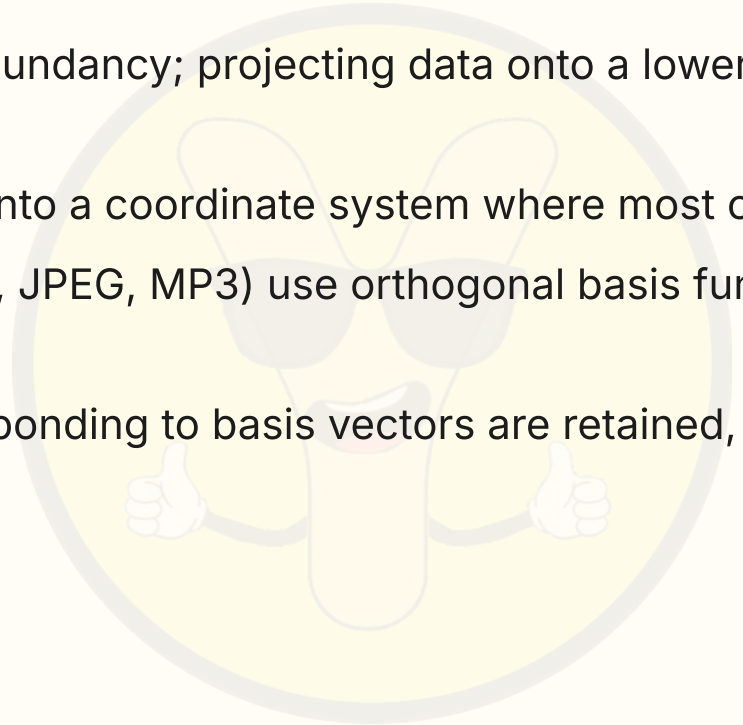
Vector spaces provide the mathematical foundation for structuring and manipulating multi-dimensional engineering data.

- Data points in multi-sensor measurements are modeled as vectors in $\mathbb{R}^n$.
- Feature matrices represent datasets where each column or row is a vector in a specific feature space.
- Linear transformations map data from high-dimensional spaces to lower-dimensional spaces (e.g., Principal Component Analysis).
- Subspaces represent constrained physical states, such as equilibrium conditions in structural mechanics.

Engineering Application: Data Compression

Vector space concepts enable efficient storage and transmission of large datasets.

- High-dimensional data contains redundancy; projecting data onto a lower-dimensional subspace reduces storage size.
- Basis change transforms raw data into a coordinate system where most coefficients are near zero.
- Image and audio compression (e.g., JPEG, MP3) use orthogonal basis functions like Discrete Cosine Transform (DCT) or wavelets.
- Only significant coefficients corresponding to basis vectors are retained, achieving high compression ratios with minimal loss.



Engineering Application: Signal Processing

Signals are treated as vectors in infinite-dimensional function spaces.

- Continuous-time signals form vector spaces over real or complex numbers under pointwise addition and scalar multiplication.
- Fourier analysis decomposes complex signals into a linear combination of orthogonal sinusoidal basis functions (sine and cosine).
- Filtering operations act as linear transformations on signal vector spaces, removing unwanted frequency components.
- Convolution in time domain maps to vector multiplication in the transformed frequency domain.

Engineering Application: Noise Reduction

Vector space projections allow separation of pure signals from additive noise.

- Measured signal $\mathbf{x}$ is modeled as sum of true signal $\mathbf{s}$ and noise $\mathbf{n}$: $\mathbf{x} = \mathbf{s} + \mathbf{n}$.
- Signal space S is defined as a subspace spanned by valid signal basis vectors.
- Orthogonal projection maps the noisy vector $\mathbf{x}$ onto the subspace S, minimizing the distance to the noise component.
- Least squares estimation uses orthogonal projections to extract optimal parameter estimates in noisy communication channels.

Exam Corner

Q1 Define a field and verify whether the set of integers $\mathbb{Z}$ forms a field under standard addition and multiplication.

Q2 State the ten axioms of a vector space. Check if the set of all polynomials of degree exactly n forms a vector space.

Q3 Define a subspace. Prove that the intersection of two subspaces of a vector space is also a subspace.

Q4 Determine whether the vectors $(1, 1, 2), (2, -1, 0), (1, 4, 6)$ in $\mathbb{R}^3$ are linearly independent.

Q5 Find a basis and the dimension of the subspace of $\mathbb{R}^4$ spanned by $(1, 2, -1, 1), (2, 4, -2, 2), (1, 3, 1, 2), (3, 7, 0, 5)$.

Q6 Explain how orthogonal projection onto a subspace is utilized in engineering noise reduction and signal processing.

Unit 5 done. You are exam-ready.

Revise the Exam corner once more, then open Unit 1 on theycotes.com. Every AKTU subject, unit by unit, free. Notes by Chirag Jain, Ycotes.

