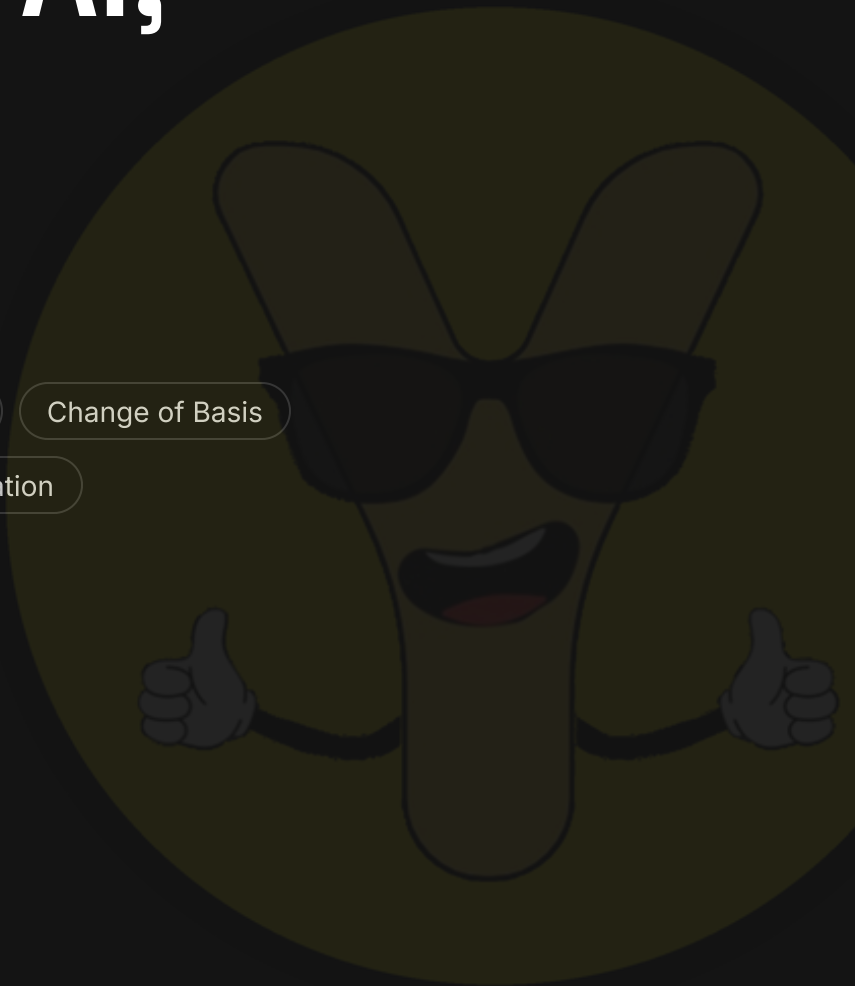
AKTU · 3RD YEAR · SEMESTER 5 · BCAI051

Mathematical Foundation AI, ML and Data Science

Unit 5 — Linear Transformations, Eigenvalues and Eigenvectors

- Linear Transformations and Matrices for Linear Transformation
- Kernel and Range of a Linear Transformation
- Change of Basis
- Definition of Eigenvalue and Eigenvector
- Diagonalization
- Symmetric Matrices and Orthogonal Diagonalization



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Unit 5 Overview: Linear Transformations and Eigen-Theory

Essential linear algebra concepts powering dimensionality reduction, matrix factorizations, and machine learning algorithms.

- Linear transformations map vectors between vector spaces while preserving vector addition and scalar multiplication.
- Matrices represent linear maps with respect to chosen bases for domain and codomain.
- Kernel and range define the null space and image space of transformation maps.
- Eigenvalues and eigenvectors capture invariant directions of linear transformations.
- Diagonalization simplifies matrix operations by decoupling system variables into independent axes.
- Orthogonal diagonalization of symmetric matrices forms the theoretical foundation of Principal Component Analysis.

Linear Transformation

LINEAR TRANSFORMATION

A mapping $T : V \to W$ between vector spaces V and W over a field F is linear if it preserves vector addition and scalar multiplication.

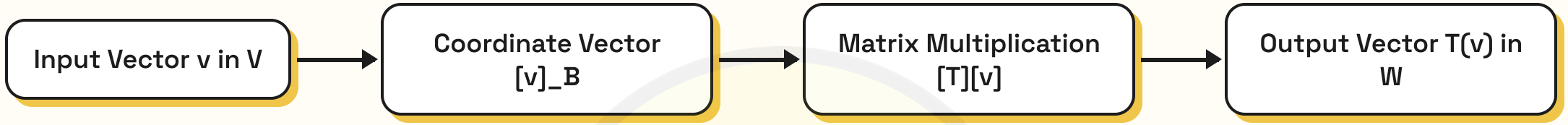
- Additivity: $T(u + v) = T(u) + T(v)$ for all $u, v \in V$.
- Homogeneity: $T(cu) = cT(u)$ for all $u \in V$ and scalar $c \in F$.
- Combined Condition: $T(cu + dv) = cT(u) + dT(v)$.
- Zero Vector Mapping: $T(0_V) = 0_W$ holds for any linear transformation.

Matrix of a Linear Transformation

Every linear transformation between finite-dimensional vector spaces can be uniquely represented as matrix multiplication.

- Let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis for domain V and C a basis for codomain W.
- Evaluate image of domain basis vectors: Compute $T(v_1), T(v_2), \ldots, T(v_n)$.
- Express each $T(v_j)$ as a linear combination of basis vectors in C.
- Form columns: The coordinate vectors of $T(v_j)$ relative to basis C form column j of matrix $[T]_{B,C}$.
- Action formula: $[T(v)]_C = [T]_{B,C}[v]_B$ holds for any vector $v \in V$.

Linear Transformation Mapping Process



Mapping process converting an input vector to an output vector via matrix representation.

Kernel and Range

KERNEL AND RANGE (IMAGE)

The Kernel $\ker(T)$ is the set of domain vectors mapped to zero, while Range $\text{Range}(T)$ is the set of output vectors achieved in the codomain.

- Kernel Definition: $\ker(T) = \{v \in V \mid T(v) = 0_W\}$, also called the Null Space.
- Range Definition: $\text{Range}(T) = \{T(v) \in W \mid v \in V\}$, also called the Image space.
- Subspace Property: $\ker(T)$ is a valid subspace of domain V, and $\text{Range}(T)$ is a subspace of codomain W.

Rank-Nullity Theorem & Properties

Fundamental relationship connecting the dimensions of kernel, range, and domain vector spaces.

- Nullity: Dimension of kernel space, written as $\text{nullity}(T) = \dim(\ker(T))$.
- Rank: Dimension of range space, written as $\text{rank}(T) = \dim(\text{Range}(T))$.
- Rank-Nullity Theorem: $\dim(V) = \text{rank}(T) + \text{nullity}(T)$.
- Injective (One-to-One): T is injective if and only if $\ker(T) = \{0\}$, meaning $\text{nullity}(T) = 0$.
- Surjective (Onto): T is surjective if $\text{rank}(T) = \dim(W)$.

Worked Example: Kernel and Range

Find the kernel, range, rank, and nullity of $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x - y, y - z)$.

1. Set $T(x, y, z) = (0, 0)$ to find kernel equations: $x - y = 0$ and $y - z = 0$.
2. Solve linear system: $x = y = z \implies (x, y, z) = x(1, 1, 1)$.
3. Kernel basis is $\{(1, 1, 1)\}$, so $\ker(T) = \text{span}\{(1, 1, 1)\}$ and $\text{nullity}(T) = 1$.
4. Apply Rank-Nullity Theorem: $\text{rank}(T) = \dim(\mathbb{R}^3) - \text{nullity}(T) = 3 - 1 = 2$.
5. Since $\text{rank}(T) = 2 = \dim(\mathbb{R}^2)$, the range space is all of $\mathbb{R}^2$.

Kernel = $\text{span}\{(1, 1, 1)\}$, Range = $\mathbb{R}^2$, Nullity = 1, Rank = 2.

Change of Basis

CHANGE OF BASIS

Change of Basis is the transformation of coordinate representations of vectors when switching from one vector space basis to another.

- Let B and B' be two different bases for an n-dimensional vector space V.
- A vector $v \in V$ has distinct coordinate vectors $[v]_B$ and $[v]_{B'}$.
- Coordinate relation: $[v]_B = P_{B \leftarrow B'}[v]_{B'}$, where P is the change-of-basis transition matrix.

Transition Matrix and Operator Transformation

Rules for converting coordinates and matrix operators under different bases.

- Transition Matrix: Columns of $P_{B \leftarrow B'}$ are coordinate vectors of B' basis elements evaluated in basis B.
- Invertibility: Transition matrix P is always non-singular, and $P_{B' \leftarrow B} = P^{-1}$.
- Operator Transformation: If A represents operator T in basis B, and A' in basis B':
- Similarity Relation: $A' = P^{-1}AP$, where $P = P_{B \leftarrow B'}$.
- Similar Matrices: Matrices A and A' share identical eigenvalues, trace, and determinant.

Worked Example: Transition Matrix

Find transition matrix P from basis $B' = \{(1,1), (1,-1)\}$ to standard basis $B = \{(1,0), (0,1)\}$ in $\mathbb{R}^2$.

1. Express first vector of B' in terms of standard basis B: $(1,1) = 1(1,0) + 1(0,1)$.
2. First column of P is $[(1,1)]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
3. Express second vector of B' in terms of standard basis B: $(1,-1) = 1(1,0) - 1(0,1)$.
4. Second column of P is $[(1,-1)]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
5. Assemble columns to get transition matrix $P_{B \leftarrow B'} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Transition Matrix $P_{B \leftarrow B'} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Eigenvalues and Eigenvectors

EIGENVALUE AND EIGENVECTOR

Given an $n \times n$ square matrix A, a non-zero vector v is an eigenvector if $Av = \lambda v$ for some scalar λ , called the eigenvalue.

- Geometric Meaning: Matrix A scales eigenvector v along its direction without rotating its span line.
- Non-zero Requirement: Eigenvector $v \neq 0$; however, eigenvalue λ can equal zero.
- Eigenspace: The set of all eigenvectors for eigenvalue λ plus zero vector, $E_\lambda = \ker(A - \lambda I)$.

Characteristic Equation and Properties

Algebraic formulation for calculating eigenvalues and fundamental mathematical properties.

- Characteristic Equation: $\det(A - \lambda I) = 0$, yielding a polynomial of degree n.
- Trace Property: Sum of all eigenvalues equals matrix trace $\sum \lambda_i = \text{tr}(A)$.
- Determinant Property: Product of all eigenvalues equals matrix determinant $\prod \lambda_i = \det(A)$.
- Invertibility: Matrix A is invertible if and only if 0 is not an eigenvalue of A.
- Triangular Matrices: Eigenvalues of diagonal or triangular matrices are simply main diagonal entries.

Worked Example: Eigenvalues and Eigenvectors

Find eigenvalues and corresponding eigenvectors for matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$.

1. Set characteristic equation: $\det(A - \lambda I) = (4 - \lambda)(3 - \lambda) - 2 = 0$.
2. Simplify polynomial: $\lambda^2 - 7\lambda + 10 = 0 \implies (\lambda - 5)(\lambda - 2) = 0$. Eigenvalues are $\lambda_1 = 5, \lambda_2 = 2$.
3. For $\lambda_1 = 5$: Solve $(A - 5I)v = 0 \implies \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \implies v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
4. For $\lambda_2 = 2$: Solve $(A - 2I)v = 0 \implies \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \implies v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Eigenvalues $\lambda = 5, 2$ with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Diagonalization of a Matrix

MATRIX DIAGONALIZATION

A square matrix A is diagonalizable if it is similar to a diagonal matrix D, meaning $A = PDP^{-1}$ for an invertible matrix P.

- Modal Matrix: Matrix P consists of linearly independent eigenvectors of A as columns.
- Diagonal Matrix: Matrix D contains corresponding eigenvalues along its main diagonal entries.
- Necessary Condition: A is diagonalizable if and only if it has n linearly independent eigenvectors.
- Sufficient Condition: An $n \times n$ matrix with n distinct eigenvalues is always diagonalizable.

Step-by-Step Diagonalization Algorithm

Systematic procedure to diagonalize an n x n square matrix.

- Step 1: Solve characteristic equation $\det(A - \lambda I) = 0$ to find all eigenvalues λ_i.
- Step 2: Solve homogeneous system $(A - \lambda_i I)v = 0$ to find basis eigenvectors for each eigenspace.
- Step 3: Check eigenvector count: If total independent eigenvectors equal n, proceed; else A is non-diagonalizable.
- Step 4: Form modal matrix $P = [v_1\ v_2\ \dots\ v_n]$ using independent eigenvectors as columns.
- Step 5: Form diagonal matrix D with matching eigenvalues along main diagonal.
- Step 6: Compute matrix inverse P^{-1} to express decomposition as $A = PDP^{-1}$.

Worked Example: Matrix Diagonalization

Diagonalize the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$.

1. Eigenvalues calculated: $\lambda_1 = 5, \lambda_2 = 2$.
2. Independent eigenvectors: $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.
3. Construct modal matrix $P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$.
4. Construct diagonal matrix $D = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$.
5. Compute inverse $P^{-1} = \frac{1}{-3} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}$.

$A = PDP^{-1}$ **with** $P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$ **and** $D = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$.

Symmetric Matrices and Spectral Theorem

Special algebraic and geometric properties of real symmetric matrices.

- Real Symmetric Matrix: A square matrix A satisfying $A^T = A$ with real entries.
- Real Eigenvalues: All eigenvalues of a real symmetric matrix are guaranteed to be real numbers.
- Orthogonal Eigenspaces: Eigenvectors corresponding to distinct eigenvalues are automatically orthogonal.
- Spectral Theorem: Every real symmetric matrix A can be orthogonally diagonalized.
- Complete Orthonormal Set: An $n \times n$ symmetric matrix always yields an orthonormal basis of $\mathbb{R}^n$.

Orthogonal Diagonalization

ORTHOGONAL DIAGONALIZATION

A matrix A is orthogonally diagonalizable if there exists an orthogonal matrix Q such that $A = QDQ^T$, where D is diagonal and $Q^T = Q^{-1}$.

- Orthogonal Matrix Property: Columns of Q form an orthonormal basis (mutually perpendicular, unit norm).
- Trivial Inverse: Inverse matrix calculation simplifies directly to transpose operation $Q^{-1} = Q^T$.
- Fundamental Theorem: Matrix A is orthogonally diagonalizable if and only if A is a real symmetric matrix.

Algorithm for Orthogonal Diagonalization

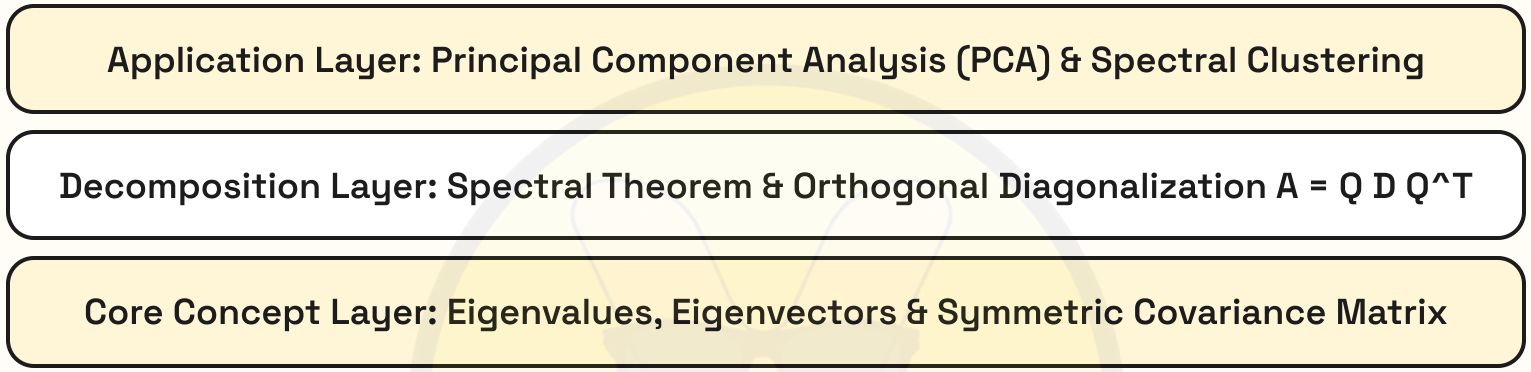
Procedure to convert a symmetric matrix into Q D Q^T form.

- Step 1: Compute all real eigenvalues of symmetric matrix A.
- Step 2: Find basis eigenvectors for each eigenspace by solving $(A - \lambda I)v = 0$.
- Step 3: Apply Gram-Schmidt process to orthogonalize eigenvectors within eigenspaces having multiplicity > 1.
- Step 4: Normalize every eigenvector to unit norm: $u_i = \frac{v_i}{\|v_i\|}$.
- Step 5: Assemble orthogonal matrix $Q = [u_1\ u_2\ \dots\ u_n]$.
- Step 6: Write diagonal matrix D matching column ordering of eigenvalues in Q.

Standard vs Orthogonal Diagonalization

Feature	Standard Diagonalization	Orthogonal Diagonalization
Matrix Requirement	Square matrix with n independent eigenvectors	Real Symmetric Matrix (A = A^T)
Transform Matrix	Invertible matrix P	Orthogonal matrix Q (Q^T = Q^-1)
Inverse Operation	Requires full matrix inversion P^-1	Simple matrix transpose Q^T
Eigenvector Basis	Linearly independent basis	Orthonormal basis
Eigenvalues	Can be real or complex numbers	Guaranteed to be real numbers
ML Application	System state transition, Matrix powers	PCA, SVD, Spectral Graph Theory

Eigen-Decomposition in Data Science



Hierarchy showing how basic eigen-theory enables practical dimensionality reduction models.

AKTU Exam Corner

Q1 Define Linear Transformation. Show that T(x, y) = (x + y, x - y) is linear and find its matrix representation in standard basis.

Q2 State and prove Rank-Nullity Theorem for linear transformations between finite dimensional vector spaces.

Q3 Define Kernel and Range of a linear transformation. Find rank and nullity of T(x, y, z) = (x + 2y, y - z).

Q4 Explain Change of Basis. Derive the relation A' = P^-1 A P for matrix representation of a linear operator.

Q5 Find all eigenvalues and corresponding eigenvectors of matrix A = [[3, 1, 4], [0, 2, 6], [0, 0, 5]].

Q6 Define matrix diagonalization. State necessary and sufficient conditions for a matrix to be diagonalizable.

Q7 Diagonalize matrix A = [[1, 2], [2, 1]] and compute A^10 using matrix diagonalization.

Q8 State Spectral Theorem for real symmetric matrices and outline the complete algorithm for orthogonal diagonalization.

Unit 5 done. You are exam-ready.

Revise the Exam corner once more, then open Unit 1 on theycotes.com. Every AKTU subject, unit by unit, free. Notes by Chirag Jain, Ycotes.

