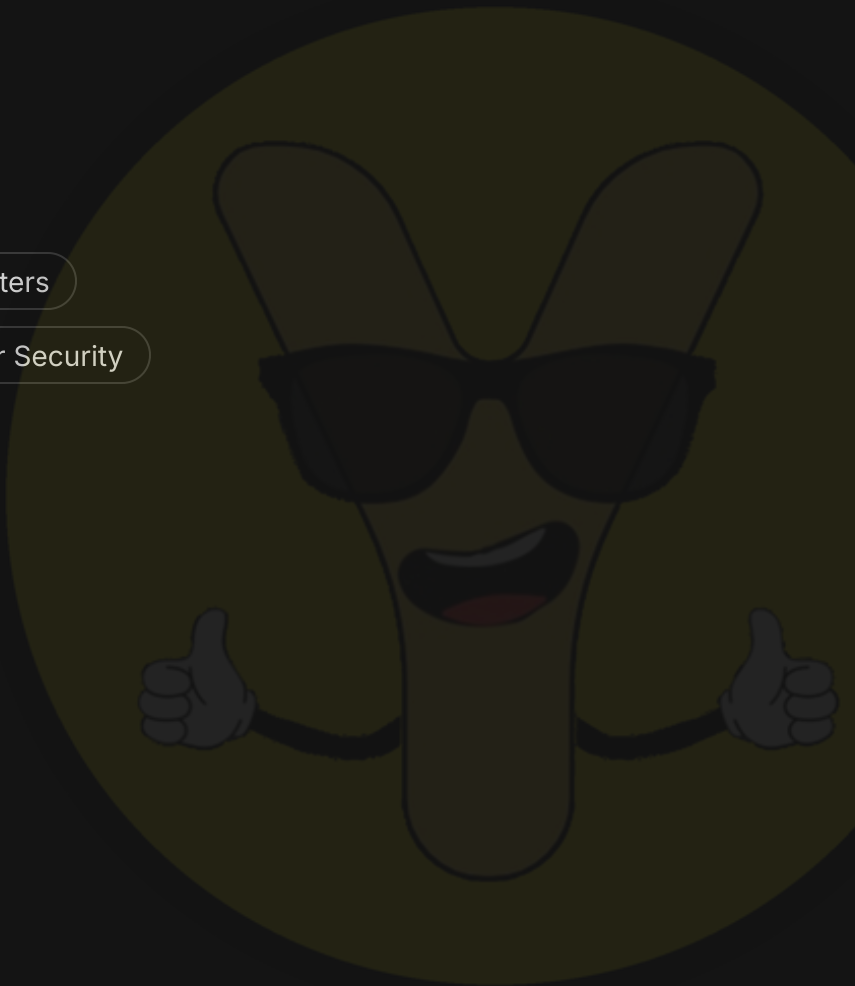
AKTU · 1ST YEAR · SEMESTER 2 · AAS203D

Numerical Methods

Unit 1 — Differential Equations

- Linear differential equation of nth order with constant coefficients
- Cauchy-Euler equation
- Solution of second order linear differential equations with variable coefficient by the method of variation of parameters
- Change of independent variable
- Normal form
- Applications: Cloud Computing, Computer Networks, Cyber Security



DECODED BY

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Unit 1 Overview: Differential Equations

Introduction to advanced analytical and numerical methods for solving ordinary differential equations in engineering applications.

- Linear differential equations with constant coefficients of order n.
- Cauchy-Euler homogenous linear differential equations with variable coefficients.
- Method of variation of parameters for second-order equations.
- Techniques involving change of independent variable and normal form.
- Real-world engineering applications in cloud computing, networks, and cyber security.

Linear ODE of nth Order

CONSTANT COEFFICIENT LINEAR ODE

An ordinary differential equation of order n where all coefficients of the dependent variable and its derivatives are constants.

- General form: $a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = X(x).$
- Here $a_0, a_1, \ldots, a_n$ are real constants and $a_0 \neq 0$.
- When $X(x) = 0$, the equation is homogeneous; otherwise, it is non-homogeneous.

Solving Homogeneous nth Order ODE

The complete solution of a homogeneous linear ODE relies heavily on its auxiliary or characteristic equation.

- Replace derivatives with operator D, where $D = \frac{d}{dx}$.
- Write auxiliary equation by replacing D with m: $f(m) = 0$.
- Real and distinct roots m_1, m_2 yield terms like $C_1 e^{m_1 x} + C_2 e^{m_2 x}$.
- Repeated roots m yield terms like $(C_1 + C_2 x)e^{mx}$.
- Complex conjugate roots $\alpha \pm i\beta$ yield $e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$.

Particular Integral (P.I.) Methods

Methods to find the particular integral y_p for non-homogeneous linear differential equations.

- General formula: $y_p = \frac{1}{f(D)} X(x)$.
- For $X(x) = e^{ax}$, replace D by a provided $f(a) \neq 0$.
- For $X(x) = \sin(ax)$ or \cos(ax), replace D^2 by $-a^2$.
- For polynomial $X(x) = x^k$, use binomial expansion of $[f(D)]^{-1}$.
- General rule for $X(x) = e^{ax}V(x)$: $\frac{1}{f(D)}[e^{ax}V(x)] = e^{ax}\frac{1}{f(D+a)}V(x)$.

Example: Constant Coefficient ODE

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$.

1. Write auxiliary equation: $m^2 - 5m + 6 = 0$.
2. Factorize to find roots: $(m-2)(m-3) = 0$, so $m = 2, 3$.
3. Complementary Function (C.F.): $y_c = C_1e^{2x} + C_2e^{3x}$.
4. Particular Integral (P.I.): $y_p = \frac{1}{D^2-5D+6}e^{4x} = \frac{e^{4x}}{4^2-5(4)+6} = \frac{e^{4x}}{2}$.

General Solution: $y = C_1e^{2x} + C_2e^{3x} + \frac{1}{2}e^{4x}$.

Cauchy-Euler Equation

HOMOGENEOUS LINEAR EQUATION WITH VARIABLE COEFFICIENTS

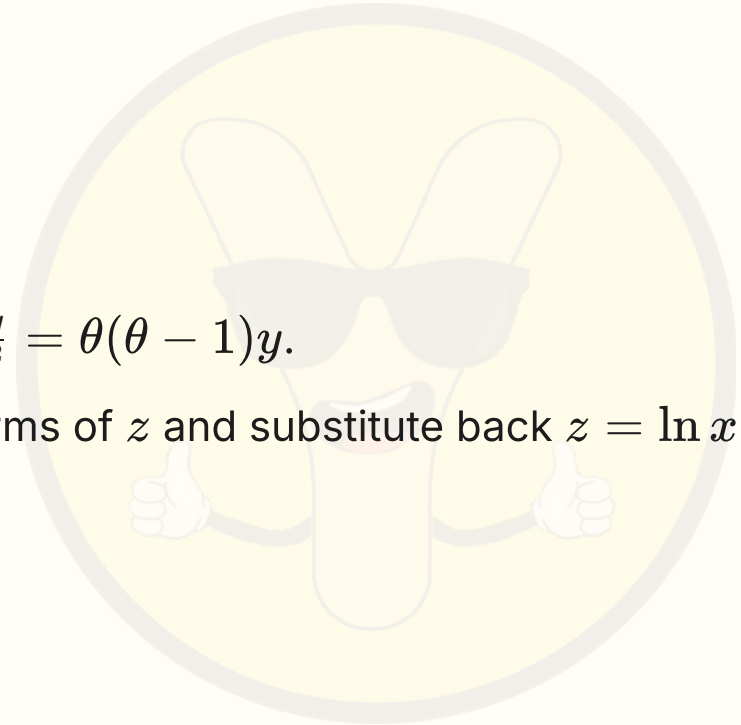
A linear differential equation where each derivative term is multiplied by a power of x matching the order of the derivative.

- General form: $x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = X(x).$
- Coefficients are linear combinations of powers of x.
- Solved by transforming into a constant coefficient equation using substitution.

Solution Method for Cauchy-Euler

Use logarithmic substitution to convert variable coefficients into constant coefficients.

- Substitute $x = e^z$ or $z = \ln x$.
- Define differential operator $\theta = \frac{dy}{dz}$.
- Transform derivatives: $x\frac{dy}{dx} = \theta y$.
- Second order transformation: $x^2\frac{d^2y}{dx^2} = \theta(\theta - 1)y$.
- Solve the resulting linear ODE in terms of z and substitute back $z = \ln x$.



Example: Cauchy-Euler Equation

Solve $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$.

1. Let $x = e^z$. Using operators $\theta = \frac{d}{dz}$, $x\frac{dy}{dx} = \theta y$, and $x^2\frac{d^2y}{dx^2} = \theta(\theta - 1)y$.
2. Substitute into equation: $[\theta(\theta - 1) - 2\theta - 4]y = 0$.
3. Simplify auxiliary equation: $\theta^2 - 3\theta - 4 = 0$, giving $(\theta - 4)(\theta + 1) = 0$.
4. Roots are $\theta = 4, -1$. Solution in z: $y = C_1e^{4z} + C_2e^{-z}$.

Substitute back $e^z = x$: $y = C_1x^4 + C_2x^{-1}$.

Method of Variation of Parameters

GENERAL METHOD FOR SECOND-ORDER ODE

A powerful technique to find the particular integral of a non-homogeneous second-order linear differential equation by varying the arbitrary constants of the complementary function.

- Standard form: $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x).$
- Requires knowing the complementary function $y_c = C_1y_1 + C_2y_2.$
- Applicable even when the coefficients $P(x)$ and $Q(x)$ are not constants.

Algorithm: Variation of Parameters

Step-by-step procedure to determine the particular integral using Wronskian.

- Step 1: Find C.F. as $y_c = C_1 y_1 + C_2 y_2$.
- Step 2: Calculate Wronskian $W = y_1 y_2' - y_2 y_1'$. Ensure $W \neq 0$.
- Step 3: Assume particular integral $y_p = A(x) y_1 + B(x) y_2$.
- Step 4: Compute parameter derivatives: $A'(x) = \frac{-y_2 R(x)}{W}$ and $B'(x) = \frac{y_1 R(x)}{W}$.
- Step 5: Integrate to find $A(x)$ and $B(x)$, then construct complete solution.

Example: Variation of Parameters

Solve $\frac{d^2y}{dx^2} + y = \csc x$ using variation of parameters.

1. C.F.: $m^2 + 1 = 0 \implies y_1 = \cos x, y_2 = \sin x$.
2. Wronskian: $W = \cos x(\cos x) - \sin x(-\sin x) = 1$.
3. Here $R(x) = \csc x$. Compute $A'(x) = \frac{-\sin x \csc x}{1} = -1 \implies A(x) = -x$.
4. Compute $B'(x) = \frac{\cos x \csc x}{1} = \cot x \implies B(x) = \ln|\sin x|$.

General Solution: $y = C_1 \cos x + C_2 \sin x - x \cos x + \sin x \ln|\sin x|$.

Change of Independent Variable

TRANSFORMATION OF INDEPENDENT VARIABLE X TO Z

A technique to simplify a second-order linear differential equation with variable coefficients by changing the independent variable from x to a carefully chosen function $z = f(x)$.

- Standard equation: $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$
- Transforms derivatives using chain rule: $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}.$
- Goal is to choose z such that the coefficient of $\frac{dy}{dz}$ becomes zero or a constant.

Working Rule for Change of Variable

Mathematical derivation for transforming second-order differential equations.

- Let $z = f(x)$. Then $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$ and $\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}$.
- Substitute derivatives into the original equation to get: $\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\left[\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right] + Qy = R$.
- To eliminate the first derivative term, set $\frac{d^2z}{dx^2} + P\frac{dz}{dx} = 0$.
- Integrate to find the transformation: $z = \int e^{-\int P dx} dx$.

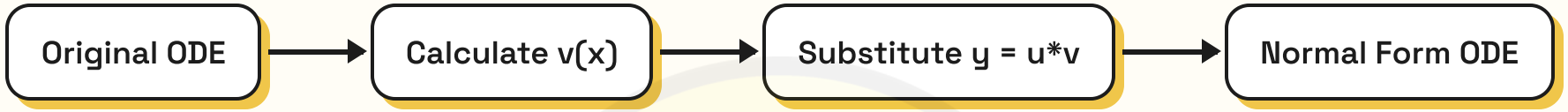
Normal Form

REMOVAL OF FIRST DERIVATIVE TERM

A transformed second-order differential equation devoid of the first derivative term, achieved by simultaneously changing both the independent and dependent variables.

- Let $y = u \cdot v$, where $v = e^{-\frac{1}{2}\int P dx}$.
- Transformed equation takes the normal form: $\frac{d^2u}{dz^2} + Iu = S$.
- Here $I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$ and $S = \frac{R}{v}$.
- Simplifies solving complex variable-coefficient differential equations.

Transformation Workflow for Normal Form

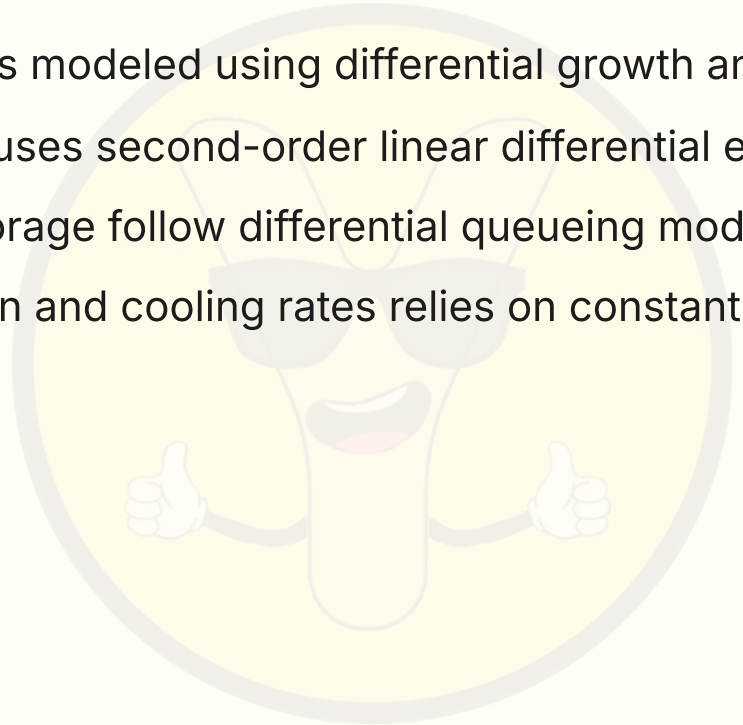


Step-by-step transformation of a linear ODE into its normal form.

Applications in Cloud Computing

Differential equations model dynamic resource allocation and load balancing in cloud data centers.

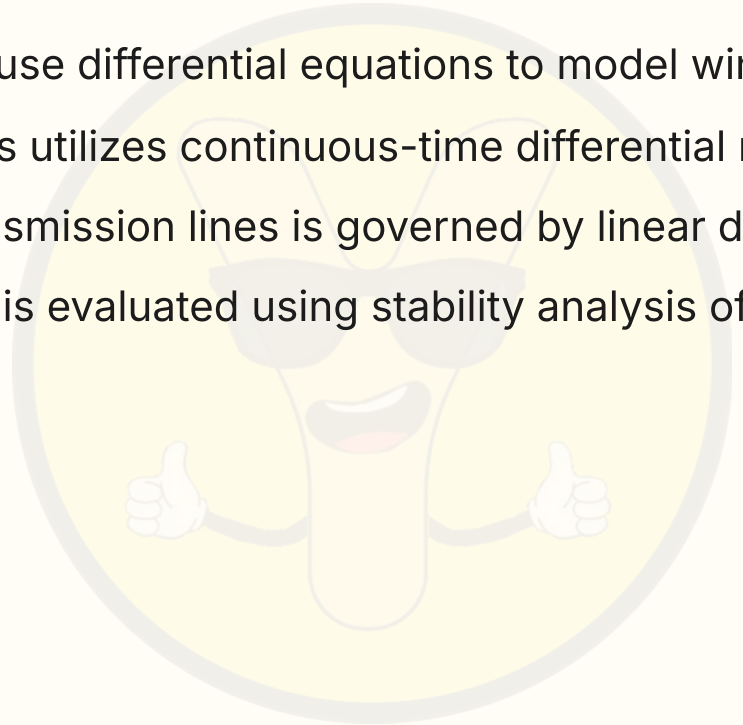
- Virtual Machine (VM) auto-scaling is modeled using differential growth and decay equations.
- Network traffic burst management uses second-order linear differential equations with damping.
- Queue length dynamics in cloud storage follow differential queueing models.
- Predicting server thermal dissipation and cooling rates relies on constant coefficient ODEs.



Applications in Computer Networks

Differential equations govern packet flow, congestion control, and signal propagation.

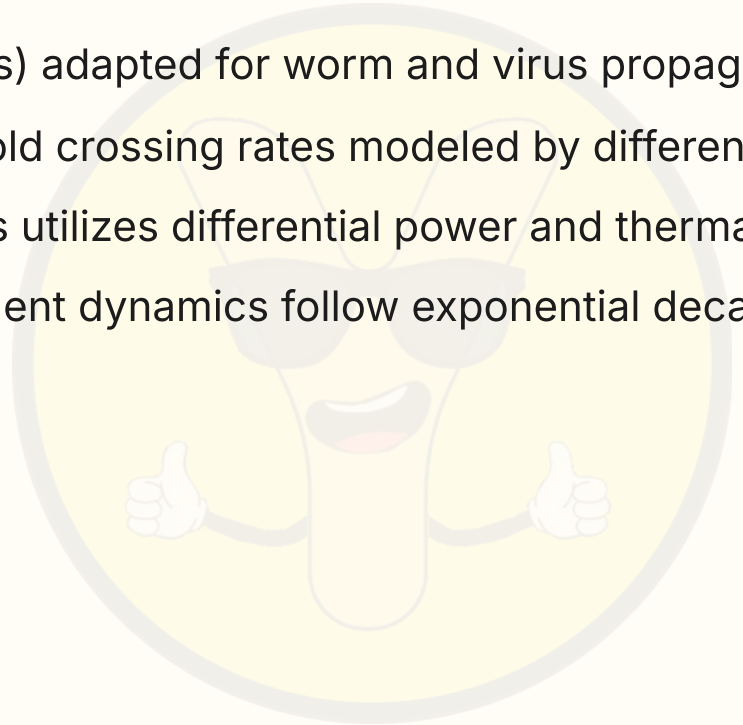
- TCP congestion control algorithms use differential equations to model window size adjustments.
- Buffer overflow estimation in routers utilizes continuous-time differential models.
- Signal attenuation in fiber optic transmission lines is governed by linear differential equations.
- Routing protocol convergence time is evaluated using stability analysis of differential systems.



Applications in Cyber Security

Differential equations analyze the spread of malware and intrusion detection system response.

- Epidemiological models (SIR models) adapted for worm and virus propagation in networks.
- DDoS attack detection uses threshold crossing rates modeled by differential equations.
- Cryptography timing attack analysis utilizes differential power and thermal models.
- System vulnerability patch deployment dynamics follow exponential decay differential equations.



Comparison of Solution Techniques

Method	Equation Type	Key Feature
Constant Coefficients	$f(D)y = X(x)$	Auxiliary equation with real/complex roots.
Cauchy-Euler	$x^n y^{(n)} + \cdots = X(x)$	Logarithmic substitution $x = e^z$.
Variation of Parameters	General 2nd Order ODE	Uses Wronskian, works for variable coefficients.
Normal Form	General 2nd Order ODE	Eliminates first derivative term via substitution.

Exam corner

Q1 Solve the linear differential equation: $(D^3 - 3D^2 + 4)y = e^{2x}$.

Q2 Solve the Cauchy-Euler equation: $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = x^2 \ln x$.

Q3 Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} + y = \tan x$.

Q4 Explain the procedure to reduce a second-order linear differential equation to normal form.

Q5 Solve by changing the independent variable: $\backslash x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = x^3 \sin x^2$.

Q6 Discuss how linear differential equations are applied in modeling network traffic or cyber security threats.

Unit 1 done. You are exam-ready.

Revise the Exam corner once more, then open Unit 2 on theycotes.com. Every AKTU subject, unit by unit, free. Notes by Chirag Jain, Ycotes.

