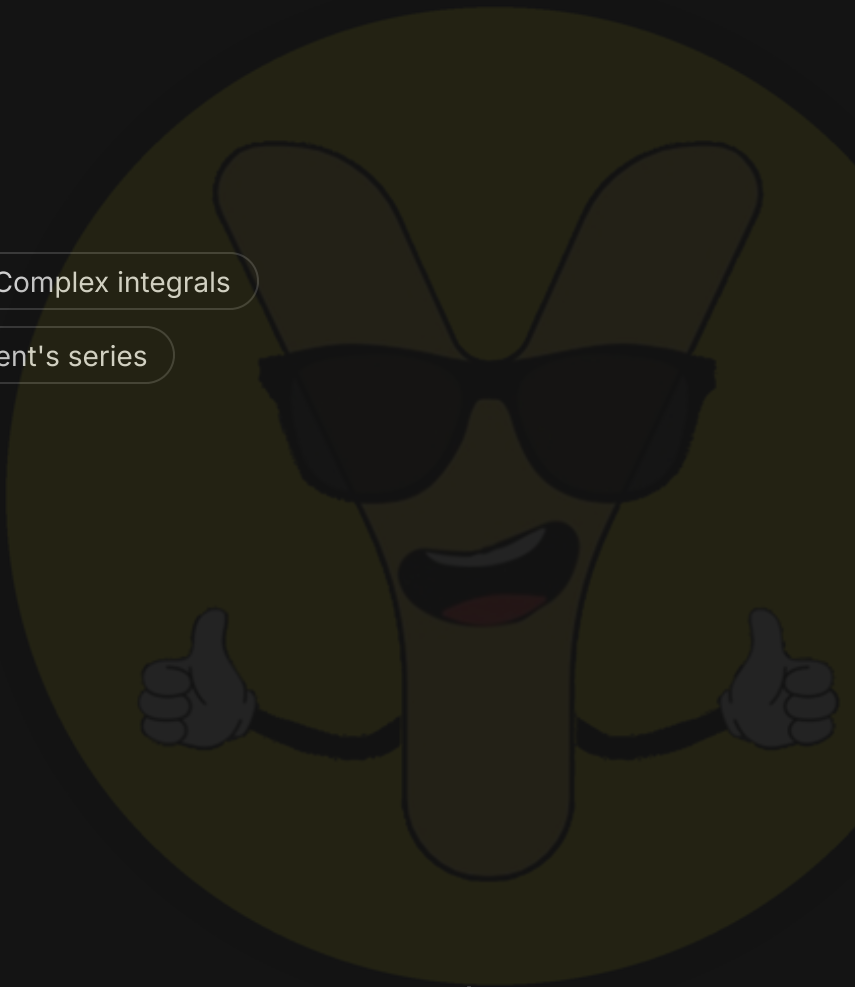
AKTU · 1ST YEAR · SEMESTER 2 · AAS203D

Numerical Methods

Unit 2 — Complex Analysis

- Functions of Complex Variables
- Limit Continuity and Differentiability
- Analytic Function
- Cauchy-Riemann Equations (Cartesian and Polar Form)
- Harmonic Function
- Milne-Thomson Method
- Complex integrals
- Cauchy Integral theorem (without proof)
- Cauchy integral formula (without proof)
- Taylor's series and Laurent's series
- Singularities
- Residue of a function
- Cauchy Residue Theorem
- Evaluating the Real definite integrals



DECODED BY

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Functions of Complex Variables

Introduction to complex mappings and variable definitions.

- A complex variable is expressed as $z = x + iy$, where $x, y \in \mathbb{R}$ and $i^2 = -1$.
- A complex function $w = f(z) = u(x, y) + iv(x, y)$ maps a subset of the complex plane to another complex plane.
- $u(x, y)$ is the real part and $v(x, y)$ is the imaginary part of the function $f(z)$.
- Unlike real functions mapped on a line, complex functions map points in a 2D plane to another 2D plane.

Limit, Continuity, and Differentiability

COMPLEX DIFFERENTIABILITY

A function f(z) is differentiable at z0 if the limit lim (z → z0) [f(z) - f(z0)] / [z - z0] exists uniquely.

- Limit: $\lim_{z \to z_0} f(z) = L$ if $f(z)$ approaches L as z approaches z_0 along any arbitrary path.
- Continuity: $f(z)$ is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.
- Differentiability requires the limit to be independent of the direction from which z_0 is approached in the complex plane.

Analytic Function

ANALYTIC / REGULAR / HOLOMORPHIC FUNCTION

A function f(z) is analytic in a domain D if it is single-valued and differentiable at every point in D.

- If a function is analytic at a specific point z_0, it must be analytic throughout a small neighborhood of z_0.
- An entire function is a function that is analytic everywhere in the finite complex plane (e.g., polynomials, e^z, $\sin z$).
- Analytic functions possess derivatives of all orders and can be expanded into Taylor series.

Cauchy-Riemann Equations (Cartesian Form)

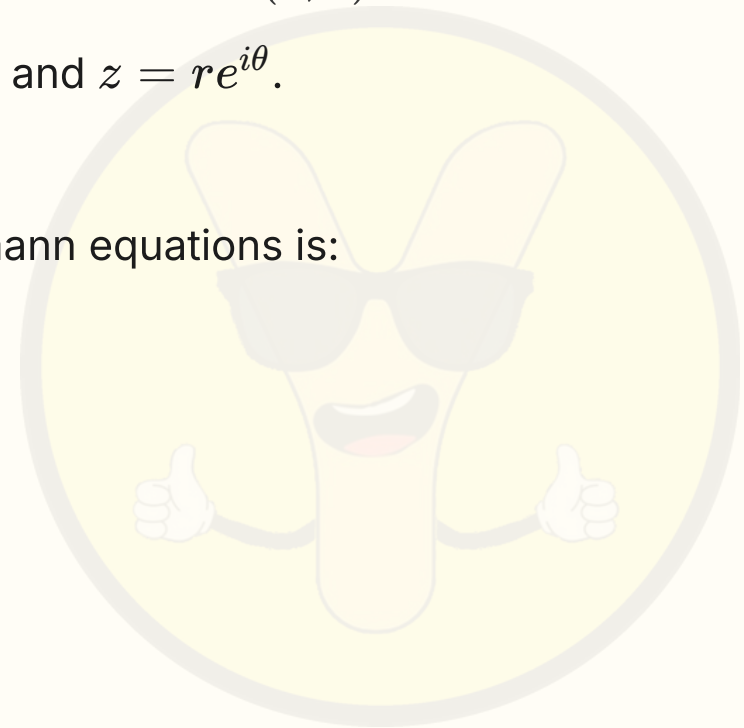
The necessary and sufficient conditions for analyticity in Cartesian coordinates.

- Let $f(z) = u(x, y) + iv(x, y)$ be differentiable at $z = x + iy$.
- The partial derivatives of u and v must satisfy the Cauchy-Riemann equations:
- $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$
- $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
- These conditions are necessary for $f(z)$ to be analytic, provided partial derivatives are continuous.

Cauchy-Riemann Equations (Polar Form)

Transforming C-R equations into polar coordinates (r, θ).

- Relations: $x = r\cos\theta$, $y = r\sin\theta$, and $z = re^{i\theta}$.
- Let $f(z) = u(r, \theta) + iv(r, \theta)$.
- The polar form of the Cauchy-Riemann equations is:
- $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$
- $\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$



Harmonic Function

HARMONIC CONJUGATE

A real-valued function of two variables is called harmonic if it has continuous partial derivatives of the first and second order and satisfies Laplace equation.

- Laplace equation in 2D: $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$
- Real and imaginary parts of an analytic function are always harmonic functions.
- If u and v satisfy C-R equations and Laplace equations, v is called the harmonic conjugate of u.

Milne-Thomson Method

An algebraic method to construct an analytic function given only its real or imaginary part.

- Step 1: Given $u(x, y)$, find partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.
- Step 2: Define $/phi_1(x, y) = \frac{\partial u}{\partial x}$ and $/phi_2(x, y) = \frac{\partial u}{\partial y}$.
- Step 3: Replace $xbyz$ and $yby0$ to get $/phi_1(z, 0)$ and $/phi_2(z, 0)$.
- Step 4: Integrate using $f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)]dz + C$.

Milne-Thomson Worked Example

Find the analytic function f(z) whose real part is u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1.

1. Calculate partials: $\phi_1(x,y) = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$ and $\phi_2(x,y) = \frac{\partial u}{\partial y} = -6xy - 6y$.
2. Substitute $x = z$ and $y = 0$: $\phi_1(z,0) = 3z^2 + 6z$ and $\phi_2(z,0) = 0$.
3. Apply Milne-Thomson formula: $f(z) = \int (3z^2 + 6z)dz$.
4. Integrate: $f(z) = z^3 + 3z^2 + C$.

f(z) = z^3 + 3z^2 + C

Complex Integrals

Integration of complex-valued functions along a directed curve in the complex plane.

- Defined as $/int_C f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(z_k)\Delta z_k$.
- Parametric evaluation: If $z = z(t)$ for $a \le t \le b$, then $/int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$.
- Path independence: If $f(z)$ is analytic in a simply connected domain, the integral depends only on endpoints, not the path.
- Properties: Linear, reversal of path changes sign of the integral, triangle inequality applies.

Cauchy Integral Theorem

CAUCHY'S THEOREM

If a function f(z) is analytic inside and on a closed contour C, then the line integral of f(z) along C is zero.

- Mathematical expression: $/oint_C f(z)dz = 0$.
- Forms the foundation of complex contour integration.
- Implies that the integral is invariant under continuous deformation of the path within the domain of analyticity.
- Extends to multiply connected domains via cuts.

Cauchy Integral Formula

CAUCHY'S INTEGRAL FORMULA

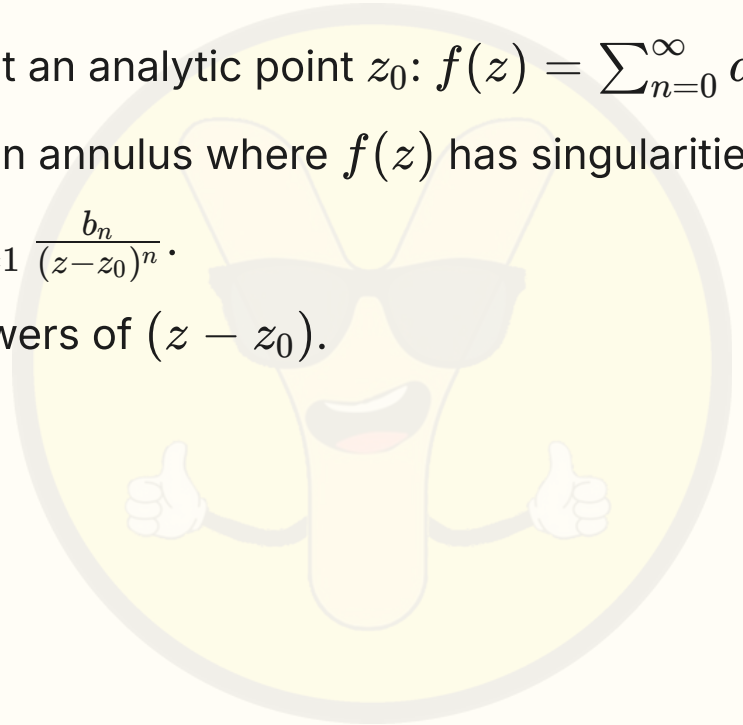
If f(z) is analytic inside and on a simple closed contour C, and z0 is any point interior to C, then f(z0) = 1/(2pi i) oint_C [f(z)/(z - z0)] dz.

- Allows calculation of the value of an analytic function inside a contour using boundary values.
- Generalized formula for derivatives: $f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz.$
- Proves that every analytic function possesses derivatives of all orders in its domain.

Taylor’s and Laurent’s Series

Representing analytic and non-analytic functions in infinite power series.

- Taylor's Series: Expands $f(z)$ about an analytic point z_0: $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, valid inside a disk.
- Laurent's Series: Expands $f(z)$ in an annulus where $f(z)$ has singularities:
- $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$.
- Principal part contains negative powers of $(z - z_0)$.



Taylor Series vs Laurent Series

Feature	Taylor Series	Laurent Series
Domain of validity	Inside a circular disk $\lvert z - z_0 \rvert < R$	Inside an annulus $r_1 < \lvert z - z_0 \rvert < r_2$
Negative powers	Does not contain negative powers	Contains negative powers of $(z - z_0)$
Singularities	Function must be analytic inside the disk	Can handle functions with singularities in the center
Application	Used for regular points	Used near isolated singularities and poles

Singularities

SINGULAR POINT

A singularity of a function f(z) is a point at which f(z) ceases to be analytic or differentiable.

- Isolated Singularity: A point z_0 where $f(z)$ is not analytic, but is analytic in a punctured disk around z_0.
- Pole of order m: When the principal part of Laurent series has a finite m terms.
- Removable Singularity: Limit exists as $z \to z_0$, but value at z_0 is undefined or mismatched.
- Essential Singularity: Principal part has infinite terms (e.g., $e^{1/z}$ at $z = 0$).

Residue of a Function

RESIDUE AT A POLE

The residue of f(z) at an isolated singularity z0 is the coefficient a_{-1} in the Laurent series expansion of f(z) about z0.

- Formula for simple pole: $\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$.
- Formula for pole of order m: $\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right]$.
- Residues simplify contour integration around complex singularities.

Cauchy Residue Theorem

CAUCHY'S RESIDUE THEOREM

If f(z) is analytic inside and on a closed contour C except at a finite number of isolated singularities inside C, then the integral is 2pi i times the sum of residues at those poles.

- Mathematical formula: $/oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k)$.
- Reduces difficult calculus integration to algebraic residue computation.
- Extremely powerful tool for evaluating complex contour integrals and real definite integrals.

Evaluating Real Definite Integrals

Evaluate $/int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$ using Contour Integration.

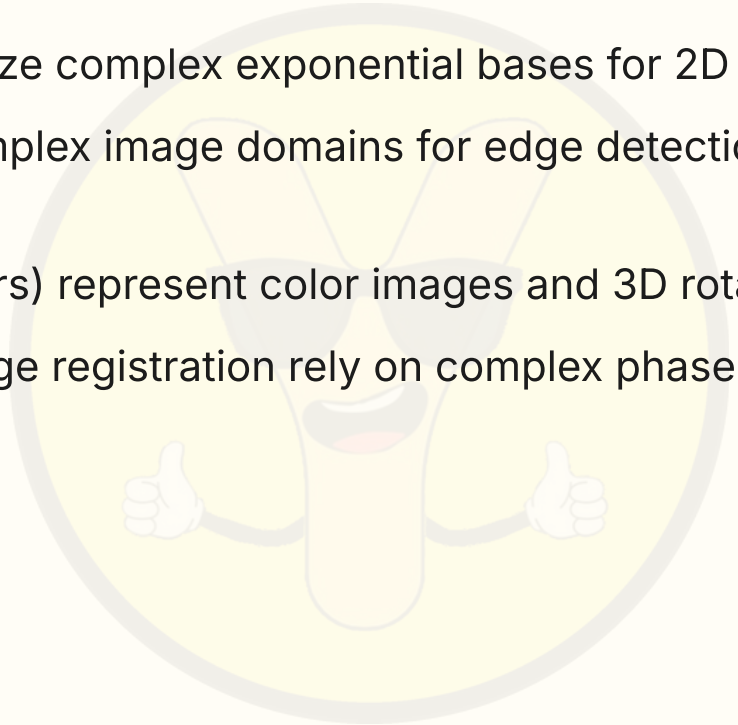
1. Substitute $z = e^{i\theta}$, so $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{z+1/z}{2}$.
2. Transform integral to a contour C (unit circle): $/oint_C \frac{dz/iz}{5+2(z+1/z)} = \frac{1}{2i}\oint_C \frac{dz}{2z^2+5z+2}$.
3. Find poles by setting denominator to zero: $2z^2+5z+2=0 \implies (2z+1)(z+2)=0$. Poles are $z=-1/2$ and $z=-2$.
4. Only $z=-1/2$ lies inside the unit circle $|z|=1$.
5. Calculate residue at $z=-1/2$: $\text{Res} = \lim_{z\to -1/2}(z+1/2)\frac{1}{2(z+1/2)(z+2)} = \frac{1}{2(3/2)} = 1/3$.
6. Apply Residue Theorem: $2\pi i \times (1/3) = \frac{2\pi i}{3}$. Multiply by $1/(2i)$ from front yields $\frac{\pi}{3}$.

pi / 3

Applications: Image Processing

Using complex analysis in digital image filtering and transformations.

- Fourier and Wavelet transforms utilize complex exponential bases for 2D image frequency analysis.
- Conformal mapping transforms complex image domains for edge detection and geometric distortion correction.
- Quaternions (hypercomplex numbers) represent color images and 3D rotations in computer vision.
- Phase correlation algorithms in image registration rely on complex phase angle properties.



Applications: Quantum Computing

Complex numbers as the fundamental mathematical framework of quantum mechanics.

- Quantum state vectors (qubits) exist in a complex Hilbert space: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $\alpha, \beta \in \mathbb{C}$.
- Normalization condition: $|\alpha|^2 + |\beta|^2 = 1$.
- Quantum gates are represented by unitary matrices with complex entries.
- Schrodinger wave equation relies fundamentally on complex numbers and differential operators.

Exam Corner

Q1 State and verify Cauchy-Riemann equations in Cartesian coordinates for an analytic function.

Q2 Show that the function $u = x^3 - 3xy^2$ is harmonic and find its harmonic conjugate using the Milne-Thomson method.

Q3 State Cauchy's Integral Theorem and evaluate $\oint_C \frac{dz}{z-a}$ where C is a circle around a.

Q4 Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in Laurent's series valid for the region $1 < |z| < 2$.

Q5 Define isolated singularities and classify the singularities of $f(z) = \frac{\sin z}{z^3}$.

Q6 State Cauchy's Residue Theorem and use it to evaluate $/oint_C \frac{z^2}{z-1} dz$ over $|z| = 2$.

Q7 Evaluate the real integral $/int_0^{2\pi} \frac{d\theta}{1+a\cos\theta}$ where $0 < a < 1$ using contour integration.

Unit 2 done. You are exam-ready.

Revise the Exam corner once more, then open Unit 3 on theycotes.com. Every AKTU subject, unit by unit, free. Notes by Chirag Jain, Ycotes.

