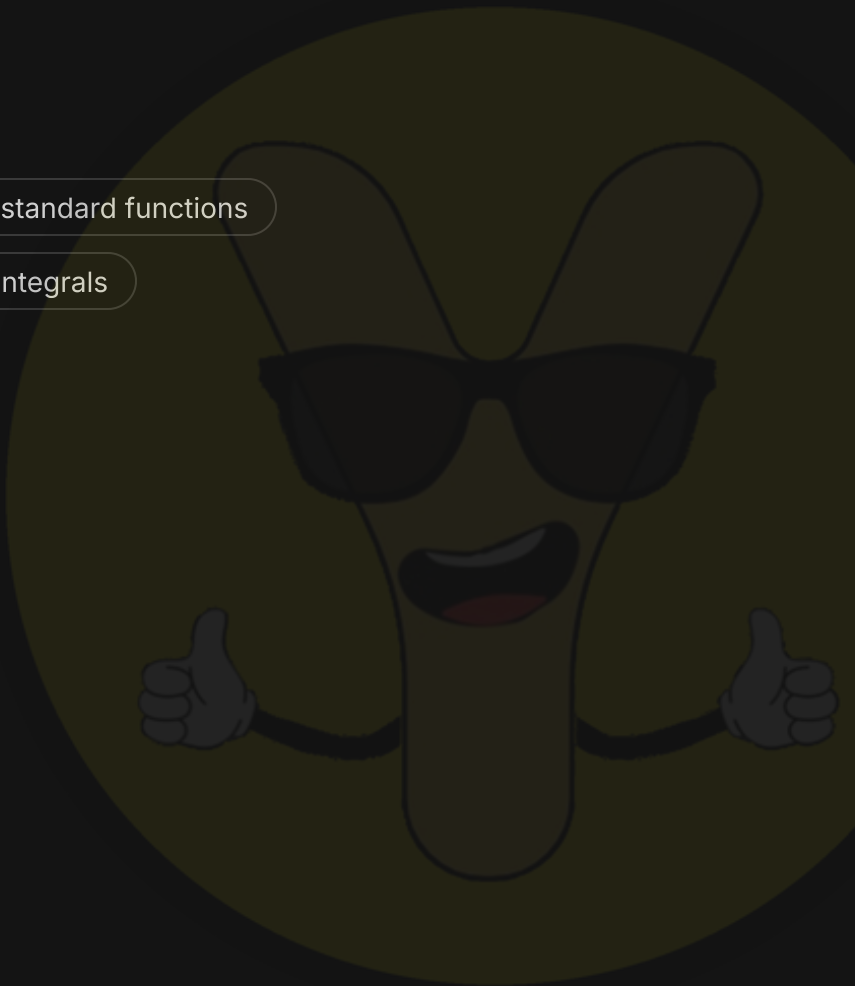
AKTU · 1ST YEAR · SEMESTER 2 · AAS203D

Numerical Methods

Unit 3 — Laplace Transform

- Definition of Laplace Transform
- Existence Conditions
- Linearity Property
- Laplace Transform of some standard functions
- I and II Shifting theorems (without proof)
- Change of Scale Property
- Laplace transform of derivatives and integrals
- Periodic Functions
- Unit Step Function
- Inverse Laplace Transform
- Partial Fraction Method
- Convolution theorem (without proof)
- Application of Laplace transform to solve ordinary differential equation



DECODED BY

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Definition of Laplace Transform

The Laplace transform maps a time-domain function into a complex frequency-domain function.

- Let $f(t)$ be a function defined for all $t \geq 0$.
- The Laplace transform of $f(t)$, denoted by $L\{f(t)\}$ or $F(s)$, is defined as:
- $F(s) = \int_0^\infty e^{-st} f(t) dt$
- Here, s is a complex parameter given by $s = \sigma + i\omega$.
- It converts complex differential equations into simpler algebraic equations.

Existence Conditions for Laplace Transform

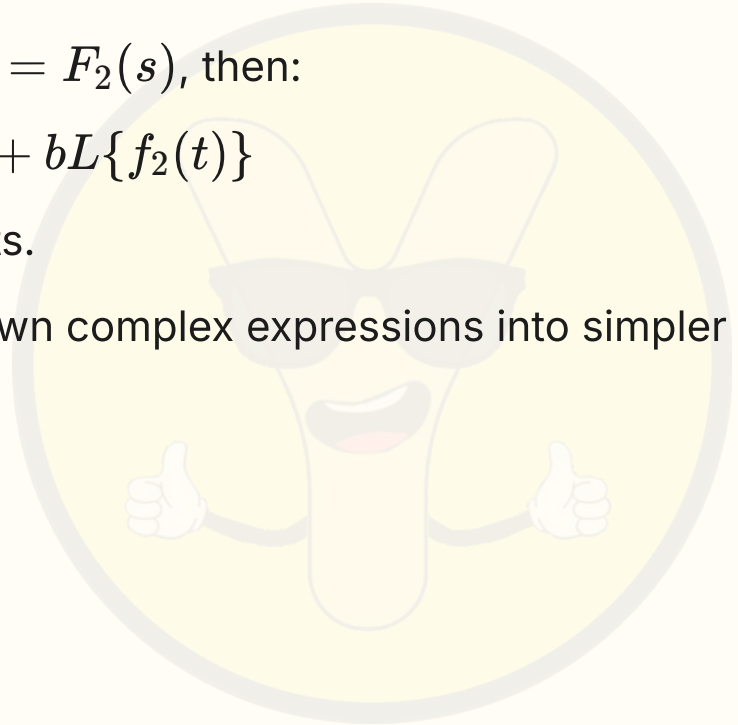
The integral defining the Laplace transform does not converge for all functions.

- Condition 1: $f(t)$ must be piecewise continuous in any finite interval $0 \leq t \leq T$.
- Condition 2: $f(t)$ must be of exponential order as $t \to \infty$.
- A function is of exponential order if there exist constants $M > 0$ and real c such that $|f(t)| \leq Me^{ct}$.
- If these conditions are met, the integral converges absolutely for $Re(s) > c$.

Linearity Property of Laplace Transform

The Laplace transform is a linear integral operator.

- If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$, then:
- $L\{af_1(t) + bf_2(t)\} = aL\{f_1(t)\} + bL\{f_2(t)\}$
- Here, a and b are arbitrary constants.
- This property allows us to break down complex expressions into simpler standard functions.



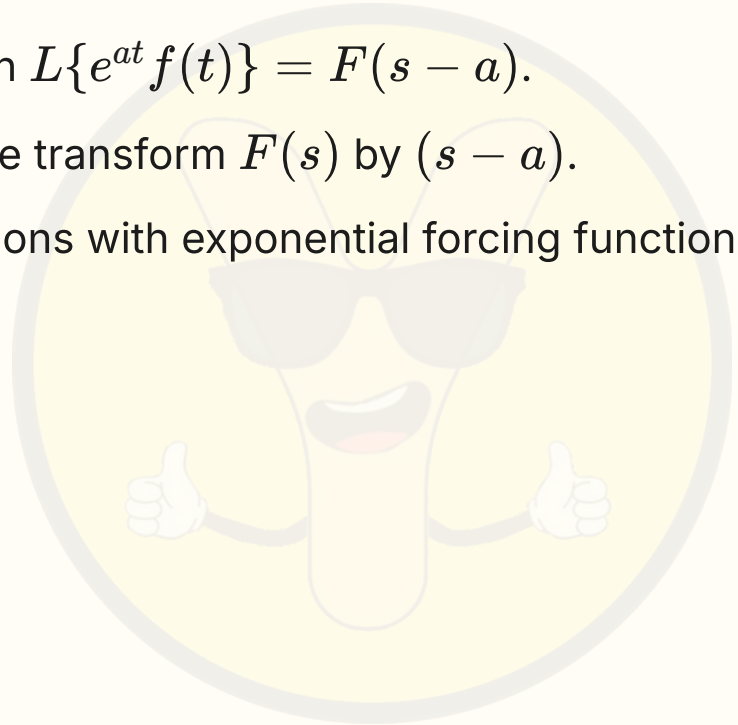
Laplace Transform of Standard Functions

Function $f(t)$	Laplace Transform $F(s)$	Region of Convergence
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n (n positive integer)	$/fracn!s^{n+1}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos(at)$	$/fracss^2 + a^2$	$s > 0$
\sinh(at)\	$\frac{a}{s^2-a^2}$	$s > \|a\|$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > \|a\|$

First Shifting Theorem

Shifts the transform along the s-axis when multiplied by an exponential.

- Statement: If $L\{f(t)\} = F(s)$, then $L\{e^{at} f(t)\} = F(s-a)$.
- Working Rule: Replace every s in the transform $F(s)$ by $(s-a)$.
- Useful for solving differential equations with exponential forcing functions.



Second Shifting Theorem

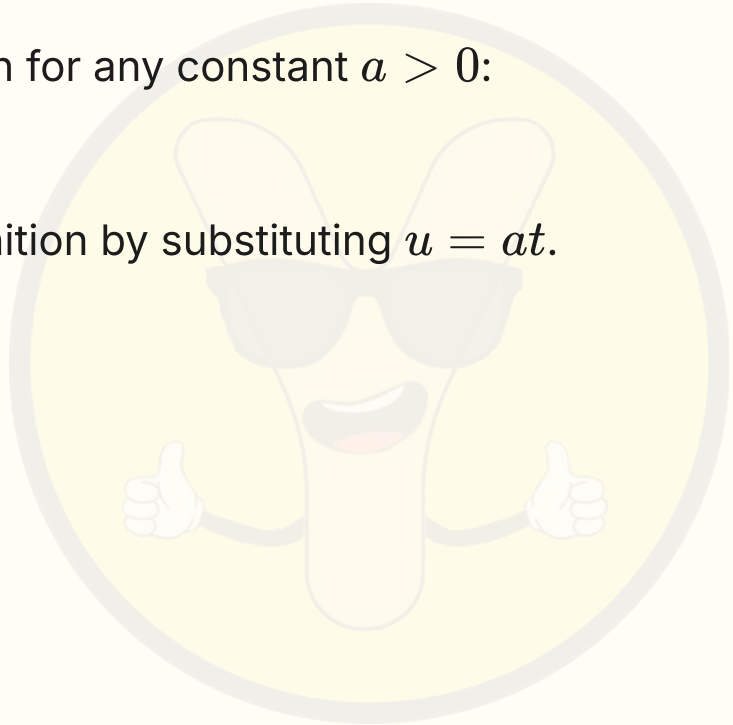
Handles functions shifted along the time axis.

- Statement: If $L\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$, then:
- $L\{g(t)\} = e^{-as}F(s)$.
- Alternatively, if $L\{f(t-a)H(t-a)\} = e^{-as}F(s)$, where $H(t)$ is the Heaviside step function.
- Crucial for analyzing delayed response systems.

Change of Scale Property

Scales the independent variable inside the transform.

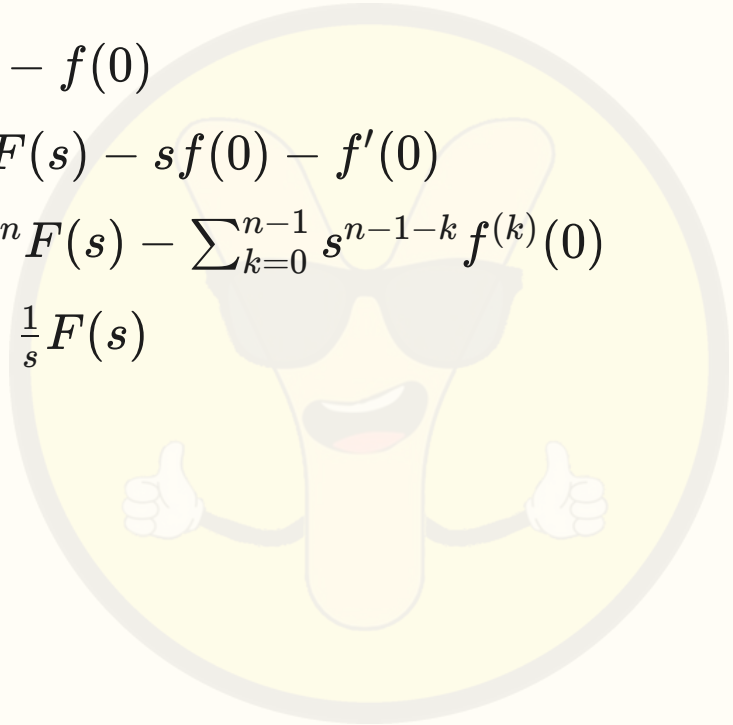
- Statement: If $L\{f(t)\} = F(s)$, then for any constant $a > 0$:
- $L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$.
- Proof follows directly from the definition by substituting $u = at$.



Laplace Transform of Derivatives and Integrals

Transforms calculus operations into algebraic operations.

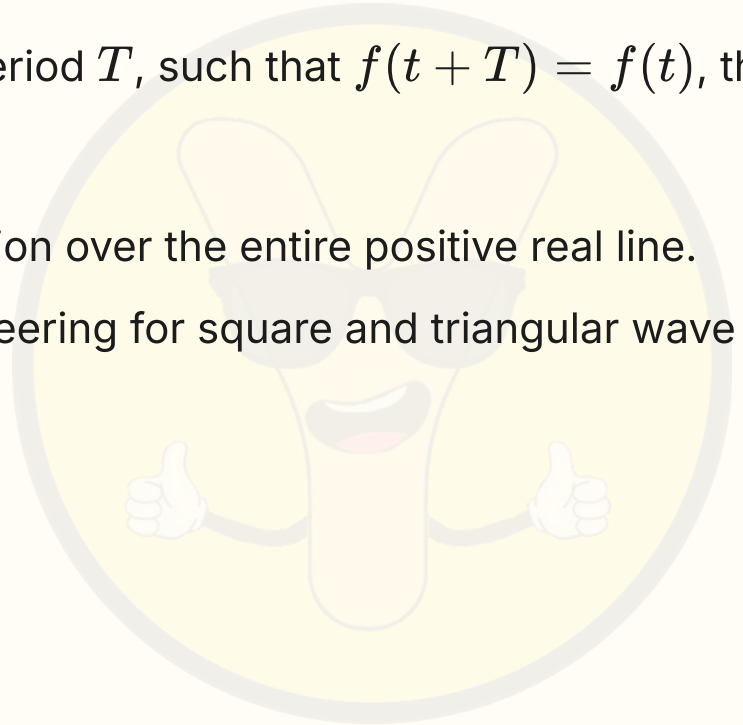
- First Derivative: $L\{f'(t)\} = sF(s) - f(0)$
- Second Derivative: $L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$
- General Derivative: $L\{f^{(n)}(t)\} = s^nF(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0)$
- Integral Property: $L\{\int_0^t f(u)du\} = \frac{1}{s}F(s)$



Laplace Transform of Periodic Functions

Simplifies transforms for repeating waveforms.

- If $f(t)$ is a periodic function with period T, such that $f(t+T) = f(t)$, then:
- $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$
- This formula avoids infinite integration over the entire positive real line.
- Extensively used in electrical engineering for square and triangular wave analysis.

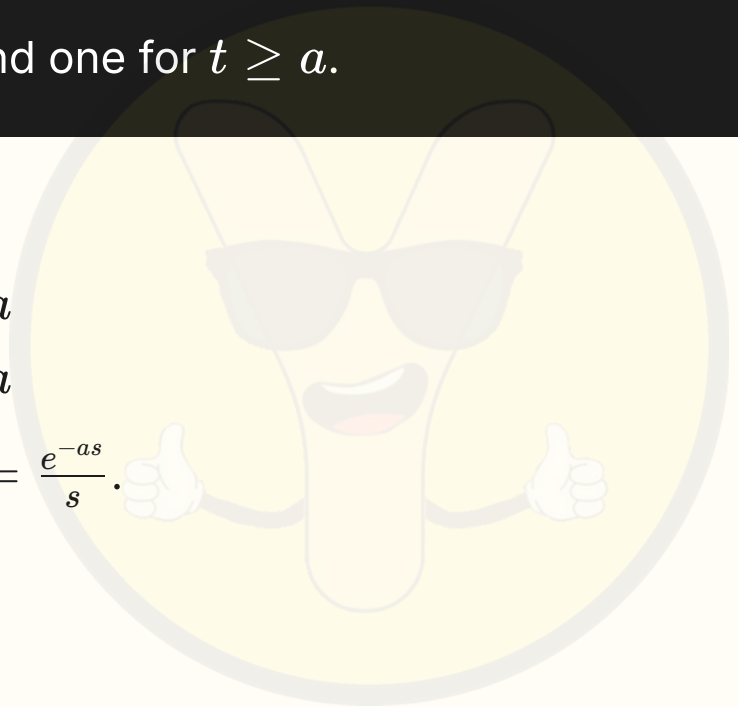


Unit Step Function (Heaviside Function)

UNIT STEP FUNCTION $H(T-A)$

A function that is zero for $t < a$ and one for $t \geq a$.

- Notation: $u(t-a)$ or $H(t-a)$.
- Definition: $H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$
- Laplace Transform: $L\{H(t-a)\} = \frac{e^{-as}}{s}$.



Inverse Laplace Transform

Recovers the time-domain function from its transform.

- If $L\{f(t)\} = F(s)$, then $f(t)$ is the inverse Laplace transform of $F(s)$.
- Notation: $f(t) = L^{-1}\{F(s)\}$.
- Linearity holds: $L^{-1}\{aF_1(s) + bF_2(s)\} = aL^{-1}\{F_1(s)\} + bL^{-1}\{F_2(s)\}$.
- Uniqueness: Lerch's theorem ensures that if two functions have the same transform, they are identical except possibly on a set of measure zero.

Partial Fraction Method for Inverse Transforms

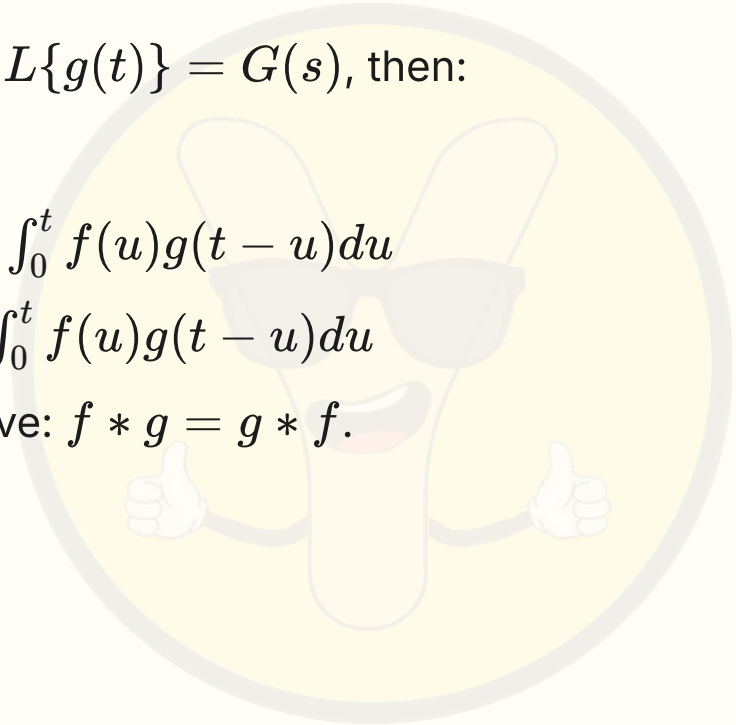
Decomposes complex rational fractions into simpler terms.

- Step 1: Ensure the degree of numerator is less than denominator; if not, use polynomial long division.
- Step 2: Factorize the denominator into linear and quadratic factors.
- Step 3: Assign constants (A, B, C) to each partial fraction.
- Step 4: Solve for constants using equating coefficients or substitution.
- Step 5: Apply standard inverse Laplace formulas to each term.

Convolution Theorem

Relates the product of two transforms to a time-domain integral.

- Statement: If $L\{f(t)\} = F(s)$ and $L\{g(t)\} = G(s)$, then:
- $L\{f(t) * g(t)\} = F(s) \cdot G(s)$
- Convolution Integral: $f(t) * g(t) = \int_0^t f(u)g(t-u)du$
- Inverse Form: $L^{-1}\{F(s)G(s)\} = \int_0^t f(u)g(t-u)du$
- Property: Convolution is commutative: $f * g = g * f$.



Solving Ordinary Differential Equations

Transforms initial value problems into algebraic solutions.

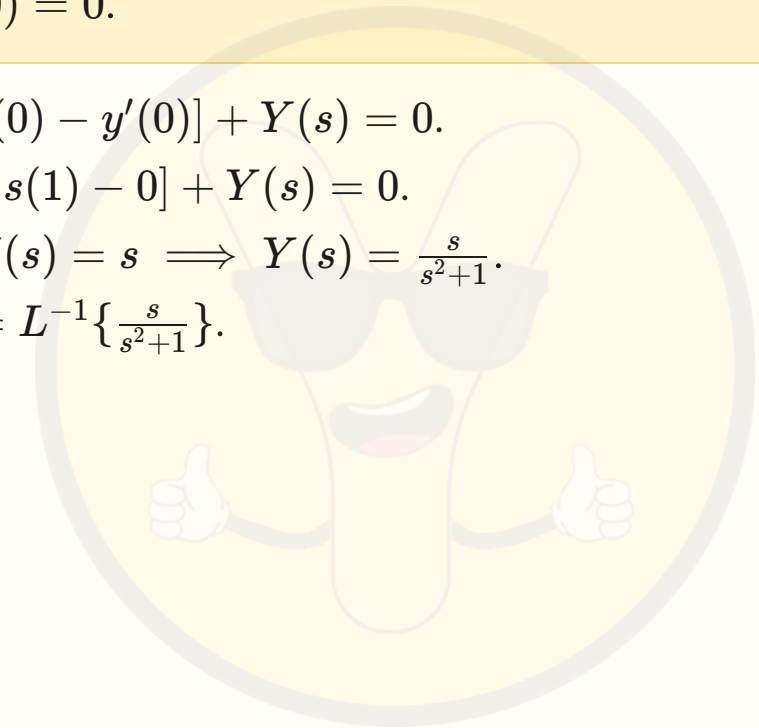
- Step 1: Take the Laplace transform of both sides of the given differential equation.
- Step 2: Substitute initial conditions $f(0), f'(0)$, etc.
- Step 3: Rearrange the algebraic equation to solve for $L\{y(t)\} = Y(s)$.
- Step 4: Apply inverse Laplace transform to obtain the solution $y(t)$.
- Advantage: Automatically incorporates initial conditions without finding general solutions and arbitrary constants first.

ODE Solution via Laplace Transform

Solve $y'' + y = 0$ given $y(0) = 1, y'(0) = 0$.

1. Take Laplace transform: $[s^2Y(s) - sy(0) - y'(0)] + Y(s) = 0$.
2. Substitute initial conditions: $[s^2Y(s) - s(1) - 0] + Y(s) = 0$.
3. Simplify and solve for $Y(s)$: $(s^2 + 1)Y(s) = s \implies Y(s) = \frac{s}{s^2+1}$.
4. Take inverse Laplace transform: $y(t) = L^{-1}\{\frac{s}{s^2+1}\}$.

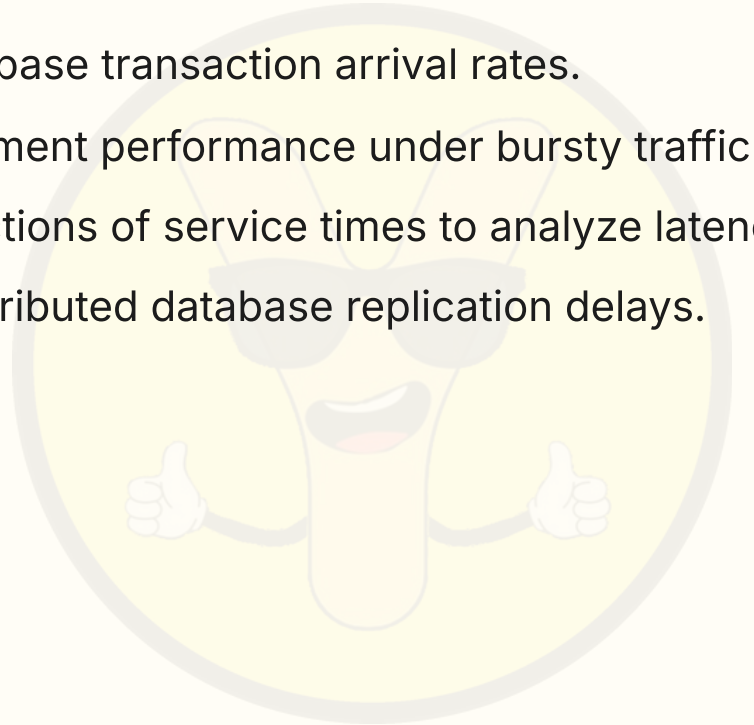
y(t) = \cos(t)



Applications in Database Systems

Laplace transforms model continuous queuing and data flow.

- Used in analytical modeling of database transaction arrival rates.
- Helps evaluate buffer pool management performance under bursty traffic.
- Transforms probability density functions of service times to analyze latency distributions.
- Assists in capacity planning for distributed database replication delays.

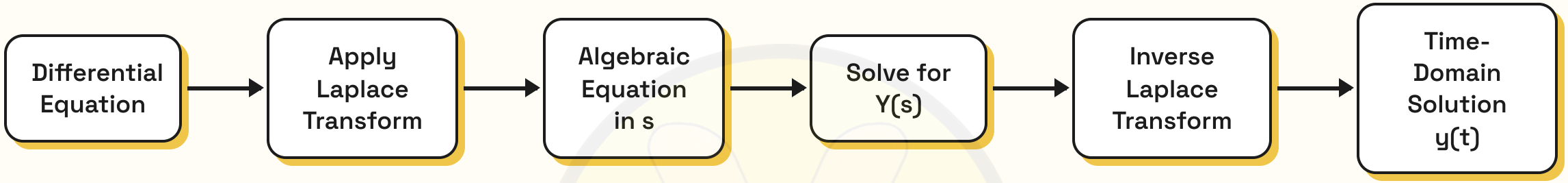


Applications in Algorithmic Modelling

Analyzes dynamic system behavior and recursive workloads.

- Applied in analyzing control systems embedded in real-time software.
- Used to solve recurrence relations in divide-and-conquer algorithm analysis via generating functions equivalent.
- Models network congestion control algorithms (like TCP packet flow dynamics).
- Evaluates stability of adaptive filtering algorithms in signal processing modules.

Laplace Transform Problem-Solving Pipeline



Standard workflow for solving linear differential equations using Laplace transforms.

Exam corner

Q1 State and prove the first and second shifting theorems for Laplace transforms with conditions.

Q2 Find the Laplace transform of the periodic square wave function of period 2T.

Q3 Evaluate $L\{te^{-3t}\sin(2t)\}$ using appropriate properties.

Q4 State the convolution theorem and use it to find $L^{-1}\{\frac{1}{(s^2+a^2)^2}\}$.

Q5 Solve the differential equation $y'' + 4y' + 4y = e^{-t}$ with $y(0) = 0, y'(0) = 0$ using Laplace transform.

Q6 Find the inverse Laplace transform of $/fracs(s+1)(s^2+4)$ using the partial fraction method.

Unit 3 done. You are exam-ready.

Revise the Exam corner once more, then open Unit 4 on theycotes.com. Every AKTU subject, unit by unit, free. Notes by Chirag Jain, Ycotes.

