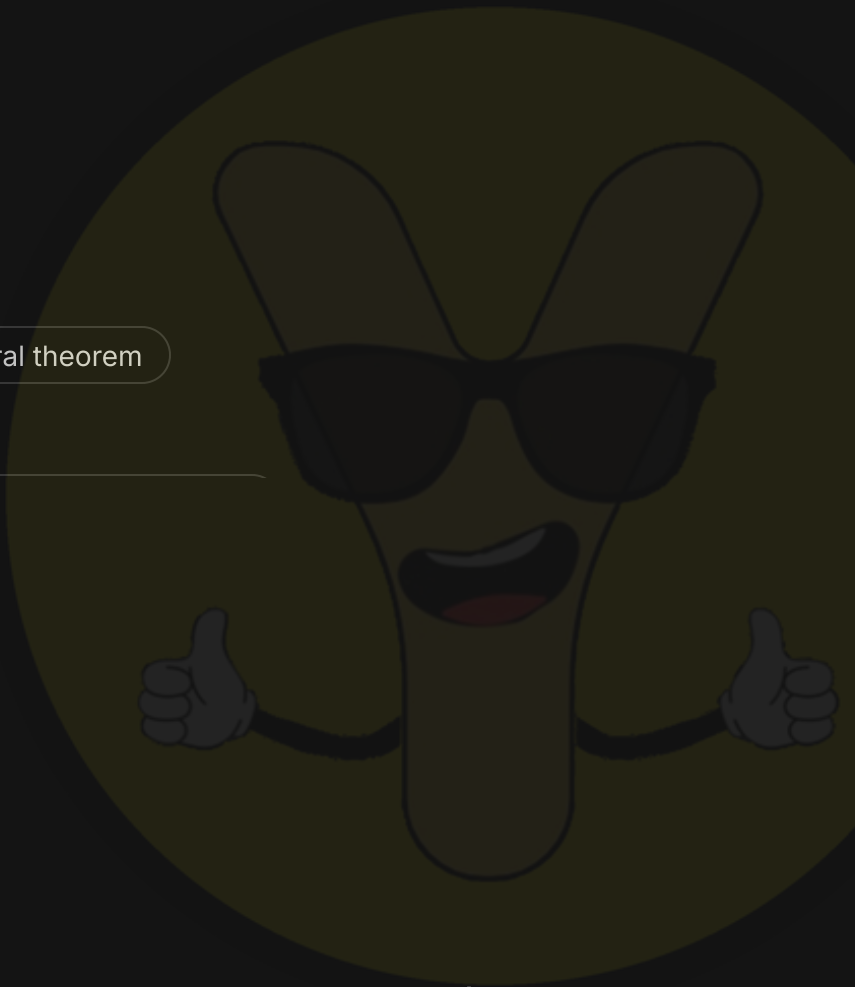
AKTU · 1ST YEAR · SEMESTER 2 · AAS203D

Numerical Methods

Unit 4 — Fourier Series and Fourier Transform

- Periodic function
- Even and odd function
- Dirichlet's Condition for the Existence of a Fourier Series
- Fourier series expansion of a function in the interval c to c+2l
- Change of Interval
- Half Range Series: Half range sine and cosine series in the interval -l to l
- Parseval's Identity
- Fourier Integral theorem
- Fourier transform
- Inverse Fourier Transform
- Finite Fourier Sine and Cosine transform



DECODED BY

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Periodic Function

PERIODIC FUNCTION

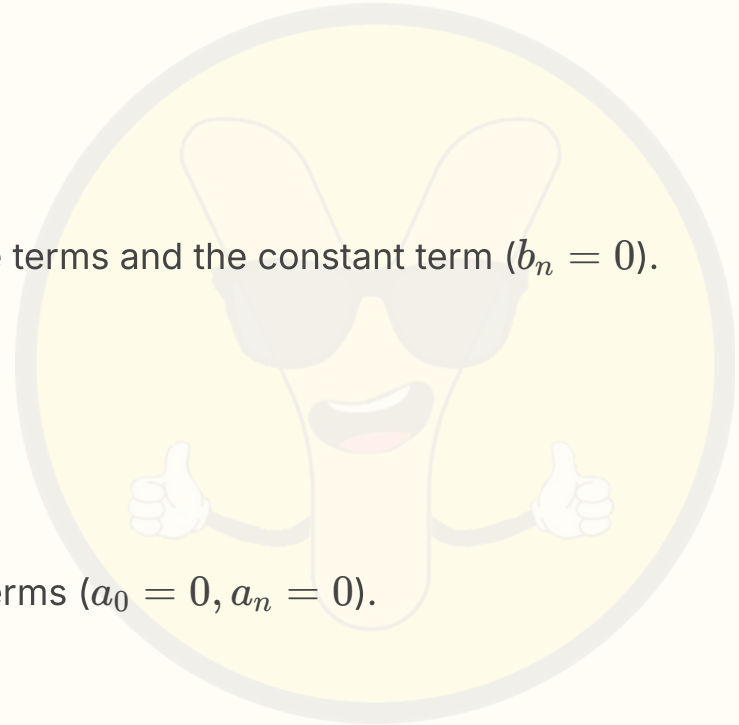
A function $f(x)$ is said to be periodic if it repeats its values at regular intervals, satisfying $f(x + T) = f(x)$ for all x, where $T > 0$ is the fundamental period.

- The smallest positive value of T is called the fundamental period.
- Examples include $\sin(x)$ and $\cos(x)$ with period 2π.
- Crucial foundation for expanding complex waveforms into simpler harmonic components.

Even and Odd Functions

Symmetry properties drastically simplify Fourier coefficient calculations.

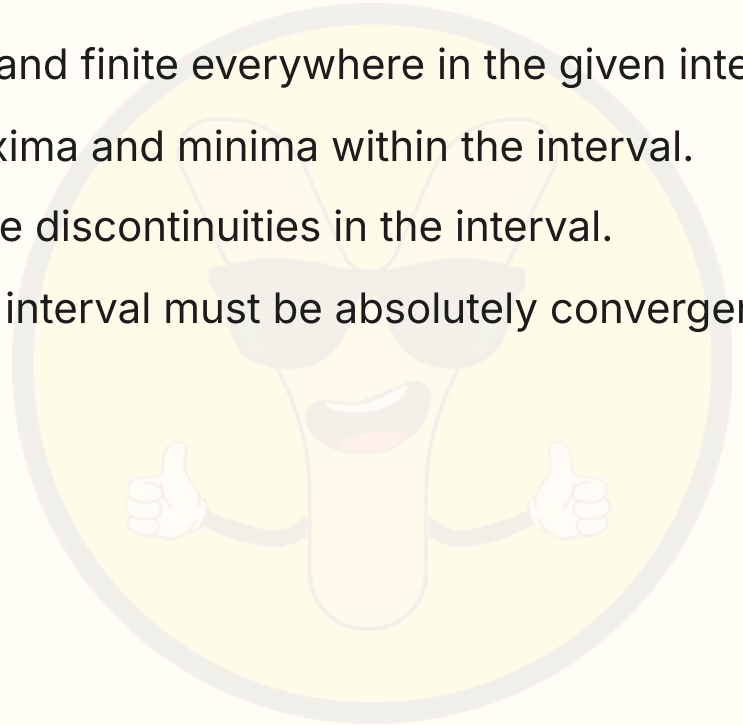
- Even Function
  - Satisfies $f(-x) = f(x)$.
  - Graph is symmetric about the y-axis.
  - Fourier expansion contains only cosine terms and the constant term ($b_n = 0$).
  - Example: $f(x) = x^2$, $f(x) = \cos(x)$.
- Odd Function
  - Satisfies $f(-x) = -f(x)$.
  - Graph is symmetric about the origin.
  - Fourier expansion contains only sine terms ($a_0 = 0, a_n = 0$).
  - Example: $f(x) = x$, $f(x) = \sin(x)$.



Dirichlet's Conditions

Conditions required for a function $f(x)$ to be expandable into a convergent Fourier series.

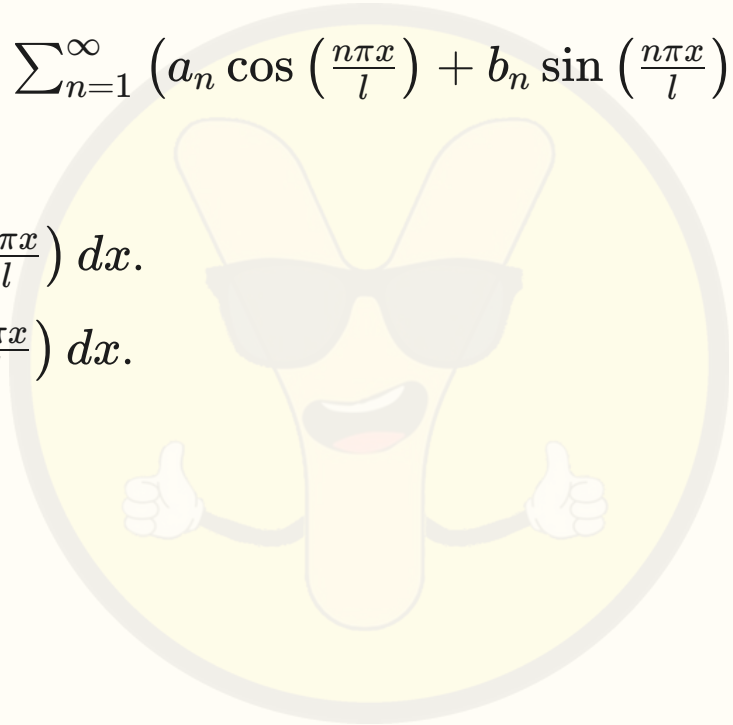
- 1. $f(x)$ is periodic, single-valued, and finite everywhere in the given interval.
- 2. $f(x)$ has a finite number of maxima and minima within the interval.
- 3. $f(x)$ has a finite number of finite discontinuities in the interval.
- 4. The integral $\int |f(x)|dx$ over the interval must be absolutely convergent.



Fourier Series Expansion on $(c, c+2l)$

Generalization of Fourier series from standard intervals to arbitrary intervals of length $2l$.

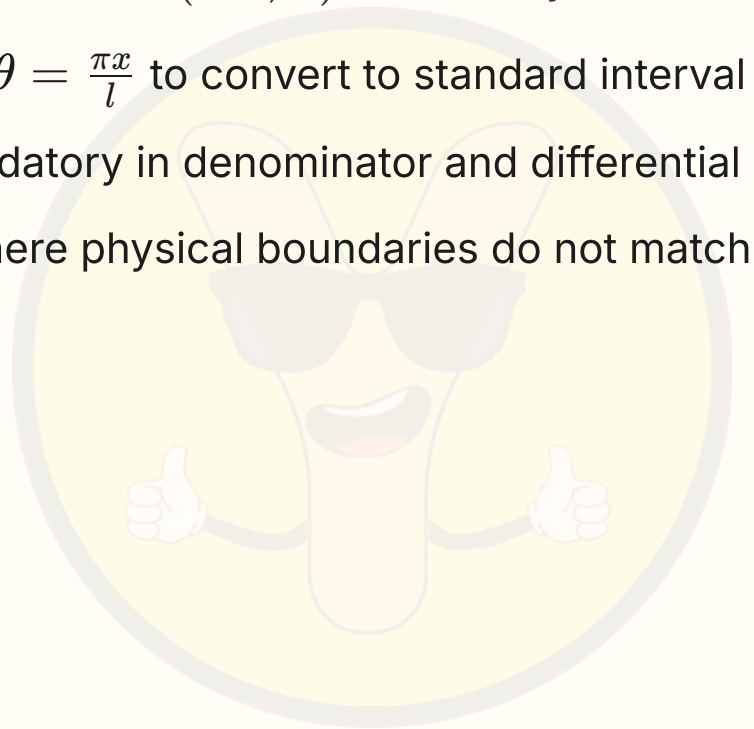
- The series is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right)$.
- Formula for a_0: $\frac{1}{l} \int_c^{c+2l} f(x)dx$.
- Formula for a_n: $\frac{1}{l} \int_c^{c+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$.
- Formula for b_n: $\frac{1}{l} \int_c^{c+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$.



Change of Interval

Techniques to transform integration limits from $(-\pi, \pi)$ to arbitrary intervals like $(-l, l)$ or $(0, 2l)$.

- If the interval is $(-l, l)$, substitute $\theta = \frac{\pi x}{l}$ to convert to standard interval $(-\pi, \pi)$.
- Scaling factor adjustments are mandatory in denominator and differential $dx = \frac{l}{\pi} d\theta$.
- Simplifies engineering problems where physical boundaries do not match mathematical constants like π.



Fourier Series Worked Example

Find the Fourier series of $f(x) = x$ in the interval $(-\pi, \pi)$.

1. Recognize $f(x) = x$ is an odd function, so $a_0 = 0$ and $a_n = 0$ for all n.
2. Calculate $b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx) dx$.
3. Use integration by parts: $b_n = \frac{2}{\pi} \left[x \left(\frac{-\cos nx}{n} \right) - (1) \left(\frac{-\sin nx}{n^2} \right) \right]_0^{\pi}$.
4. Evaluate limits: \(b_n = \frac{2}{\pi} \left(-\frac{\pi \cos n\pi}{n} ", "f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)\).

$$f(x) = 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \dots\right)$$

Half Range Series

Expanding a function defined only in half the interval by extending it as even or odd.

- Half Range Cosine Series: Assumes an even extension, so all $b_n = 0$.
- Formula: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$ where $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$.
- Half Range Sine Series: Assumes an odd extension, so $a_0 = 0$ and all $a_n = 0$.
- Formula: $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$.

Parseval's Identity

PARSEVAL'S THEOREM

Relates the sum of the squared coefficients of a Fourier series to the integral of the squared function over a period.

- Mathematical expression: $\frac{1}{l}\int_{-l}^{l}[f(x)]^2dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$.
- Represents conservation of energy in signal processing.
- Used extensively to evaluate infinite series sums using Fourier coefficients.

Fourier Integral Theorem

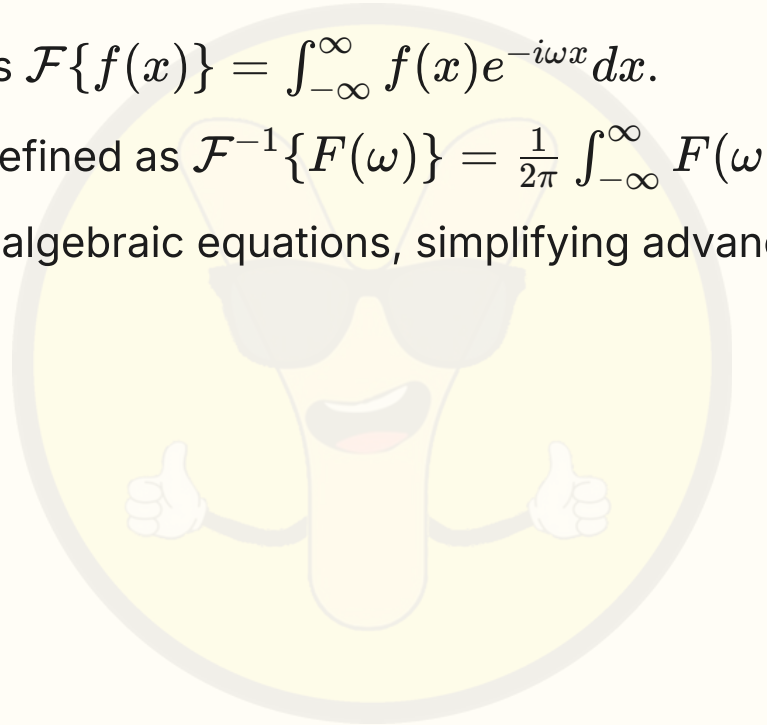
Extends Fourier series representation from periodic functions to non-periodic functions defined on infinite intervals.

- As the period $l \to \infty$, the discrete spectrum of frequencies transitions into a continuous spectrum.
- Theorem statement: $f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos(\alpha(t-x)) dt d\alpha$.
- Can also be expressed in complex exponential form: $f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{i\alpha x} \left[\int_{-\infty}^\infty f(t) e^{-i\alpha t} dt \right] d\alpha$.

Fourier Transform and Inverse

Mapping functions between time/spatial domain and frequency domain.

- Fourier Transform $F(\omega)$: Defined as $\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$.
- Inverse Fourier Transform $f(x)$: Defined as $\mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega$.
- Converts differential equations into algebraic equations, simplifying advanced engineering solutions.

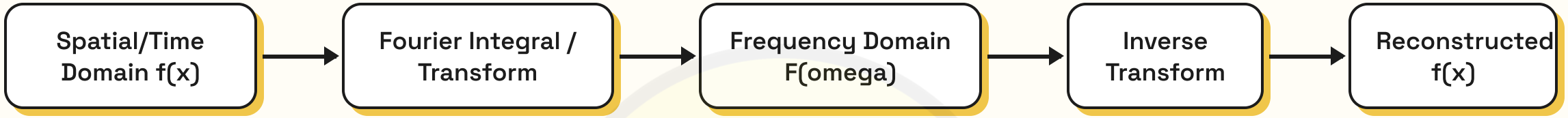


Finite Fourier Sine and Cosine Transforms

Specialized transforms used when the domain of the function is finite, typically $[0, l]$.

- Finite Fourier Sine Transform: $F_s(n) = \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$.
- Finite Fourier Cosine Transform: $F_c(n) = \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$.
- Inverse transforms reconstruct the original function using corresponding sine or cosine series summations.
- Extremely useful for solving boundary value problems and partial differential equations on finite domains.

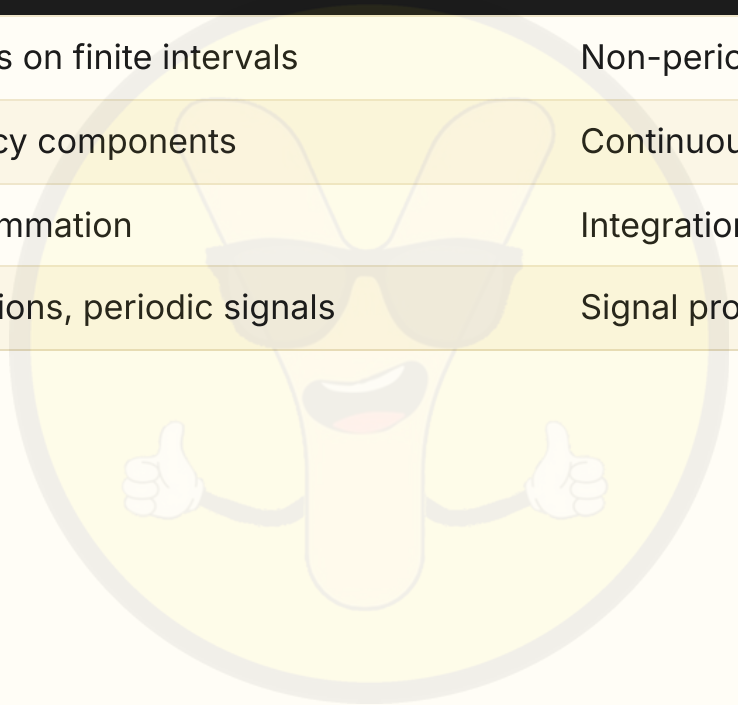
Fourier Domain Transformation Pipeline



The systematic pipeline of moving data across domains for analysis and filtering.

Comparison of Series and Transforms

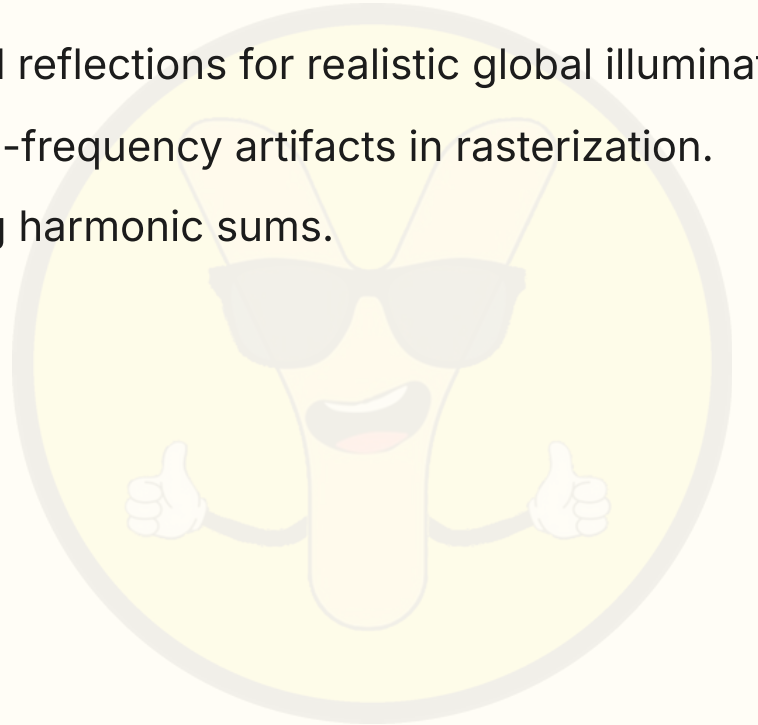
Feature	Fourier Series	Fourier Transform
Domain	Periodic functions on finite intervals	Non-periodic functions on infinite intervals
Spectrum	Discrete frequency components	Continuous frequency spectrum
Mathematical Tool	Trigonometric summation	Integration over infinite limits
Primary Use	Differential equations, periodic signals	Signal processing, image analysis, filtering



Application: Computer Graphics

Fourier techniques power advanced rendering and geometric processing.

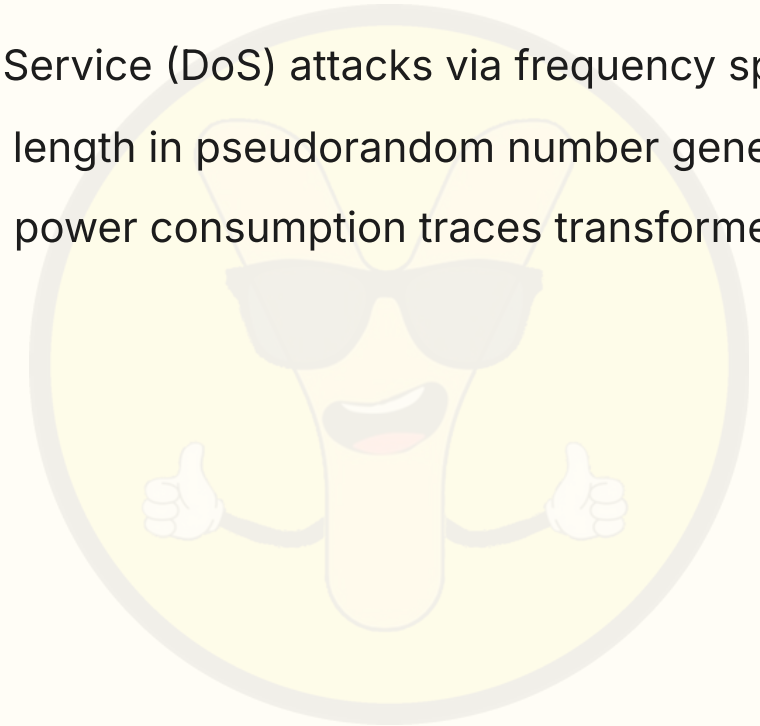
- Frequency analysis of shadows and reflections for realistic global illumination.
- Antialiasing algorithms remove high-frequency artifacts in rasterization.
- Procedural texture generation using harmonic sums.



Application: Cyber Security

Signal analysis aids in network traffic monitoring and cryptography.

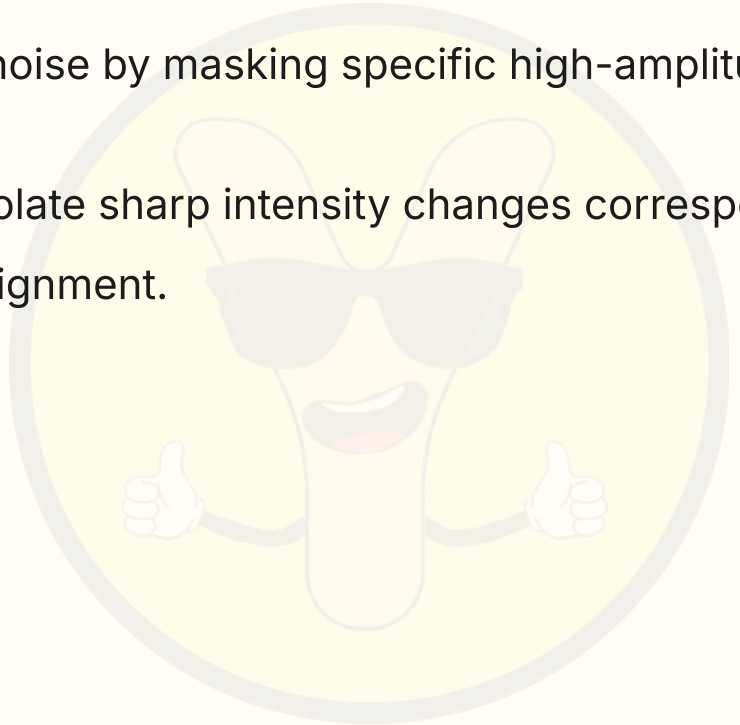
- Detecting anomalies and Denial of Service (DoS) attacks via frequency spectrum analysis of network traffic.
- Evaluating randomness and period length in pseudorandom number generators.
- Side-channel attack analysis using power consumption traces transformed into frequency domains.



Application: Digital Image Processing

Images are 2D spatial matrices that benefit immensely from 2D Fourier transforms.

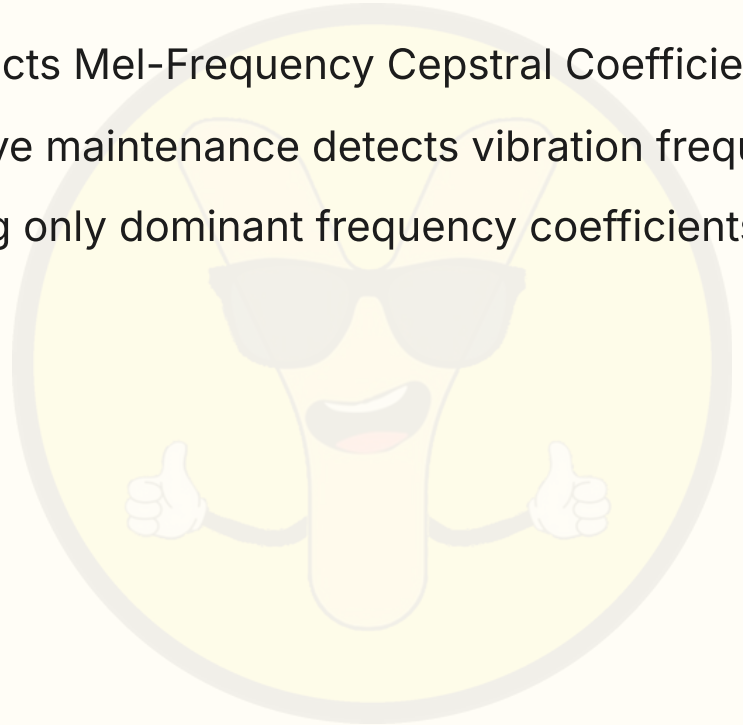
- Image filtering: Removing periodic noise by masking specific high-amplitude frequencies in the frequency domain.
- Edge detection: High-pass filters isolate sharp intensity changes corresponding to boundaries.
- Image registration and geometric alignment.



Application: Machine Learning Feature Extraction

Converting raw time-series or sensor data into descriptive frequency-domain features.

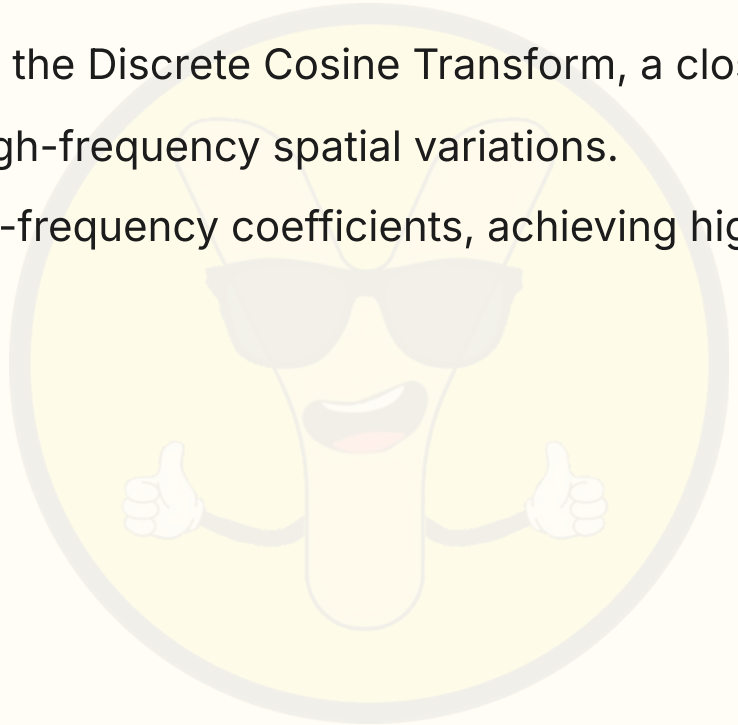
- Audio and speech recognition extracts Mel-Frequency Cepstral Coefficients derived from Fourier spectra.
- Sensor data analysis in IoT predictive maintenance detects vibration frequencies of failing machinery.
- Reduces dimensionality by retaining only dominant frequency coefficients.



Application: Image Compression

Reducing storage requirements by discarding imperceptible high-frequency data.

- JPEG compression relies heavily on the Discrete Cosine Transform, a close variant of Fourier series.
- Human eyes are less sensitive to high-frequency spatial variations.
- Quantization matrices zero out high-frequency coefficients, achieving high compression ratios with minimal visual loss.



Exam corner

Q1 State Dirichlet's conditions for the existence of a Fourier series. Expand $f(x) = x^2$ in $(-\pi, \pi)$.

Q2 Find the half-range sine series for $f(x) = \pi - x$ in the interval $(0, \pi)$.

Q3 Obtain the Fourier series for $f(x) = |x|$ in the interval $(-\pi, \pi)$ and deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}$.

Q4 State and prove Parseval's identity for Fourier series.

Q5 Find the Fourier transform of $f(x) = e^{-|x|}$ for $-\infty < x < \infty$.

Q6 Explain the difference between Fourier series and Fourier transform with examples.

Q7 Discuss how finite Fourier transforms are applied in solving partial differential equations.

Q8 Write short notes on the role of Fourier transforms in Digital Image Processing and Image Compression.

Unit 4 done. You are exam-ready.

Revise the Exam corner once more, then open Unit 5 on theycotes.com. Every AKTU subject, unit by unit, free. Notes by Chirag Jain, Ycotes.

